Competitive facility location problem

Competitive facility location problem is a generalization of the simple plant location problem. Two competing sides called the Leader and the Follower sequentially open their facilities in order to maximize their profit obtaining from clients serving. Every client patronises only one facility, which is the most preferable for the client. The problem is to determine the set of facilities to be opened by the Leader to maximize his profit provided that the Follower captures some clients.

We use the following notations:
$I = \{1, \dots, m\}$ — the set of facilities (places for facilities openning);
$J = \{1, \dots, n\}$ — the set of clients;
$f_i, i \in I$ — fixed cost of the i–th Leader's facility openning;
$g_i, i \in I$ — fixed cost of the i–th Follower's facility openning;
$p_{ij}, i \in I, j \in J$ — profit of the i–th facility from the j–th client serving;
$r_{ij}, i \in I, j \in J$ — matrix of clients preferences. For $i, s \in I$ and $j \in J$ enequality $r_{ij} < r_{sj}$ holds if and only if facility i is more preferable than facility s for client j. For each $j \in J$ values $r_{ij}, i \in I$ are pairwise distinct and taken from the set $\{0, \dots, m-1\}$.

In the model we use following variables:
$x_i, i \in I$, which will be equal to one if facility i is opened by the Leader and zero otherwise;
$x_{ij}, i \in I, j \in J$, which will be equal to one if the facility i opened by the Leader is the most preferrable for client j among all Leader's facilities and zero otherwise;
$z_i, i \in I$, which will be equal to one if facility i is opened by the Follower and zero otherwise;
$z_{ij}, i \in I, j \in J$, which will be equal to one if the facility i opened by the Follower is the most preferrable for client j among all opened facilities and zero otherwise.

Using introduced variables the competitive facility location problem is written as follows:

$$\max_{(x_i),(x_{ij})} \left(-\sum_{i \in I} f_i x_i + \sum_{j \in J} \left(\sum_{i \in I} p_{ij} x_{ij} \right) \left(1 - \sum_{i \in I} \tilde{z}_{ij} \right) \right) \tag{1}$$

$$x_i + \sum_{k \in I | r_{kj} > r_{ij}} x_{kj} \le 1, \quad i \in I, j \in J; \tag{2}$$

$$x_i \ge x_{ij}, \quad i \in I, j \in J; \tag{3}$$

$$x_i, x_{ij} \in \{0, 1\}, \quad i \in I, j \in J; \tag{4}$$

Where $((\tilde{z}_i), (\tilde{z}_{ij}))$ is optimal solution of the problem (5)–(8);

$$\max_{(z_i),(z_{ij})} \left(-\sum_{i \in I} g_i z_i + \sum_{j \in J} \sum_{i \in I} q_{ij} z_{ij} \right) \tag{5}$$

$$x_i + z_i + \sum_{k \in I | r_{kj} > r_{ij}} z_{kj} \leq 1, \quad i \in I, j \in J; \tag{6}$$

$$z_i \geq z_{ij}, \quad i \in I, j \in J; \tag{7}$$

$$z_i, z_{ij} \in \{0, 1\}, \quad i \in I, j \in J; \tag{8}$$

The bi–level problem (1)–(8) consists of the upper level problem (1)–(4) and the lower level problem (5)–(8)

Let $X = ((x_i), x_{ij})$ be a feasible solution of the upper level problem and $\tilde{Z} = ((\tilde{z}_i), (\tilde{z}_{ij}))$ be an optimal solution of the lower level problem. When the pair $(X, \tilde{Z})$ is called a *feasible solution* of the problem (1)–(8). For the feasible solution $(X, \tilde{Z})$ the value of the objective function (1) we will denote $L(X, \tilde{Z})$

The feasible solution $(X, \bar{Z})$ is called *noncooperative feasible solution* if the enequality $L(X, \bar{Z}) \leq L(X, \tilde{Z})$ holds for all optimal solutions $\tilde{Z}$ of the lower level problem.

Noncoolperative feasible solution $(X^*, \bar{Z}^*)$ is called an *optimal solution* of the problem (1)–(8) if the enequality $L(X^*, \bar{Z}^*) \geq L(X, \bar{Z})$ holds for all noncooperative solutions $(X, \bar{Z})$.

Input parameters of the model are:
the number m;
the number n;
the matrix $(r_{ij}), i = 1, \ldots, m; j = 1, \ldots, n$;
the matrix $(p_{ij}), i = 1, \ldots, m; j = 1, \ldots, n$;
the vector $(f_i), i = 1, \ldots, m$;
the vector $(g_i), i = 1, \ldots, m$.