A Hybrid Memetic Algorithm for the Competitive p-Median Problem [⋆]

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Abstract: In the competitive p-median problem, two decision makers, the leader and the follower, compete to attract clients from a given market. The leader opens his facilities, anticipating that the follower will react to the decision by opening his/her own facilities. The leader and the follower try to maximize their own profits. This is the Stackelberg game. We present it as a linear bilevel 0–1 problem. It is known that the problem is Σ_2^P-complete. We develop a hybrid memetic algorithm for it where the follower problem is solved by commercial software. To obtain an upper bound for this maximization problem, we reformulate the bilevel problem as a single level mixed integer programming problem with exponential number of constraints and variables. Removing some of them, we get the desired upper bound. For finding an appropriate family of constraints and variables, we use metaheuristics again. As a result, we get near optimal solutions for the bilevel problem with an a posteriori bound for deviation from the global optimum. Computational results for Euclidian test instances are discussed.

Keywords: Memetic algorithm, $(r \mid p)$–centroid problem, competitive location, bilevel programming.

1. INTRODUCTION

Much effort in the discrete location theory has aimed at developing insights concerning the classical models with one decision maker only. In this paper, we study a competitive model with two noncooperative decision markers: the leader and the follower. They compete to attract clients from a given market and wish to maximize their own profits. In fact, this is the noncooperative Stackelberg game. Following Hakimi (Hakimi (1990)), we call it *the $(r|p)$-centroid problem.* We present the game as a 0–1 linear bilevel problem and develop a hybrid memetic algorithm for finding near optimal solutions. The Probabilistic Tabu Search heuristic (PTS) is used to improve each element of the population. To compute the leader profit, we have to solve the follower problem by commercial software.

In order to get an upper bound for this maximization problem, we rewrite the bilevel problem as a single level mixed integer linear program with exponential number of constraints and variables. Similar approach is suggested in (Rodriguez and Perez (2008)) for partial enumeration. If we extract a small family of constraints and variables, we get an upper bound. The PTS heuristic is used for generating the family. For comparison, we describe the upper bound from (Kochetov et al. (2009)). Computational experiments for Euclidean test instances from the benchmark library *Discrete Location Problems* (http://math.nsc.ru/AP/benchmarks/Competitive/p_med_comp_eng.html) indicate that new upper bound dominates the previous one and the difference between the upper and lower bounds is small.

The paper is organized as follows. In Section 2, we present a mathematical statement of the problem. In Section 3, the lower bounds are discussed. We present three lower bounds; two of them are quite simple, while the last one is based on the metaheuristic approach for the bilevel mathematical formulation. In Section 4, two upper bounds are described. These bounds are obtained as the optimal solution for the mixed integer linear programming problems. In Section 5, computational results are reported. Finally, in Section 6 we point out some conclusions and promising directions for future research.

2. THE COMPETITIVE P-MEDIAN PROBLEM

We are given a set $I = \{1, \dots, m\}$ of facilities and a set $J = \{1, \dots, n\}$ of clients. A matrix (g_{ij}) defines the distances between clients and facilities. If client j is serviced from a facility, he gives a profit $w_j > 0$. The leader and the follower open facilities. First, the leader opens p facilities. Later on, the follower opens r facilities. Each client chooses the closest open facility. We need to find p facilities for the leader to maximize his profit.

Let us present this game as a linear 0–1 bilevel programming problem. Define the decision variables:

$$x_i = \begin{cases} 1 & \text{if facility } i \text{ is opened by the leader,} \\ 0, & \text{otherwise,} \end{cases}$$

$$y_i = \begin{cases} 1 & \text{if facility } i \text{ is opened by the follower,} \\ 0, & \text{otherwise,} \end{cases}$$

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$$z_j = \begin{cases} 1 & \text{if client } j \text{ is serviced by the leader,} \\ 0 & \text{if client } j \text{ is serviced by the follower.} \end{cases}$$

For a given solution x, we can define the set of facilities which allow to capture client j by the follower:

$$I_j(x) = \{i \in I \,|\, g_{ij} < \min_{l \in I}(g_{lj} \,|\, x_l = 1)\}, \quad j \in J.$$

Note that we consider *conservative* clients. If a client has the same distances to the closest leader and the closest follower facilities, he prefers the leader facility. So, the follower never opens a facility at a site where the leader has opened a facility. Other models of the client behavior can be found in (Drenzer (2004)).

Now the model can be written as a linear 0–1 bilevel programming problem:

$$\max_x \sum_{j \in J} w_j z_j^*(x) \qquad (1)$$

s.t.

$$\sum_{i \in I} x_i = p, \qquad (2)$$

$$x_i \in \{0,1\}, \quad i \in I, \qquad (3)$$

where $z_j^*(x),\ y_i^*(x)$ is the optimal solution of the follower problem:

$$\max_{y,z} \sum_{j \in J} w_j(1 - z_j) \qquad (4)$$

s.t.

$$\sum_{i \in I} y_i = r, \qquad (5)$$

$$1 - z_j \le \sum_{i \in I_j(x)} y_i, \quad j \in J, \qquad (6)$$

$$x_i + y_i \le 1, \quad i \in I, \qquad (7)$$

$$y_i, z_j \in \{0,1\}, \quad i \in I, j \in J. \qquad (8)$$

The objective function (1) defines the total profit of the leader. Equation (2) guarantees that the leader opens exactly p facilities. The objective function (4) defines the total profit of the follower. Equation (5) guarantees that the follower opens exactly r facilities. Constraint (6) determines the values of z by the decision variables y of the follower. Constraint (7) admits to open a facility by at most one decision maker. Note that the sum of the leader and the follower profits is a constant, $\sum_{j \in J} w_j$. So, we deal with the $(r \mid p)$–centroid problem which is Σ_2^P-complete (Noltemeier et al. (2007)).

Note that the optimal value for the problem does not change if we replace 0–1 variables z_j by continuous variables $0 \le z_j \le 1$. So, we deal with the mixed integer linear programming problem for the follower.

Matrix (g_{ij}) is used to define the set $I_j(x)$. In fact, we can use not only the distances, but any kind of preferences for the clients. For example, a client may prefer a facility with the minimal traveling and waiting time rather than with the minimal distance. In (Vasil'ev et al. (2009)) the facility location models with general client preferences are studied. With almost no change, we can use the preferences in the definition of set $I_j(x)$. However, we preserve the distances for simplicity.

3. LOWER BOUNDS

An arbitrary feasible solution of the problem (1)–(8) produces a lower bound. For a given solution x, we have to solve the *maximum coverage problem* (4)–(8) to get a feasible solution. It is an NP–hard problem (Hakimi (1990)). We use the commercial CPLEX software for it. So, the rest of this section is devoted to describing various strategies for the selection of solution x.

The first and the simplest strategy is to ignore the follower. The leader opens facilities to minimize the total distance between clients and his facilities. He wishes to service all clients and solves the classical p-median problem. We use an optimal solution of this problem as solution x for the lower bound. This strategy is not so bad despite ignoring the follower. As we can see in our computational experiments, the leader losses more than half of the market, but we can improve the lower bound by a few percentages only.

The second strategy is more sophisticated. The leader anticipates that the follower will react to his decision. So, the $(p+r)$ facilities will be opened. According to the second strategy, the leader solves the $(p+r)$-median problem and opens p most profitable facilities. As we will see below, this strategy is not perfect.

The most powerful strategy is to solve the problem (1)–(8). We develop a Hybrid Memetic Algorithm (HMA) where tabu search approach (Glover and Laguna (1997)) is used to improve the elements of the population. Now we present the general framework of our method.

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Hybrid Memetic Algorithm
1 Generate an initial population of the leader solutions.
2 Repeat until the stopping condition is met:
    2.1 Select two solutions $x^1$, $x^2$ from the population.
    2.2 Create a solution $x$ by a recombination of $x^1$, $x^2$.
    2.3 Apply random modification to $x$ .
    2.4 Improve the solution by the PTS heuristic.
    2.5 Update the population.
3 Return the best found solution.
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We use the total number of iterations 2.1–2.5 as the stopping condition.

3.1 Initial population

To create a high quality initial population at Step 1 of the framework, we apply the standard local improvement algorithm with random starting points. The well–known Swap neighborhood for the p-median problem (Mladenović et al. (2007)) is used for the improvements. Remind that we have to solve the maximum coverage problem in order to compute the objective function value for each element of the neighborhood. The CPLEX software is very effective for this type of problems (Caprara et al. (2000)). However, the Swap neighborhood contains $p(m-p)$ elements. It is a time–consuming procedure. To reduce the running time, we use the first improvement pivoting rule, randomization of the neighborhood, and solve the linear programming relaxations to estimate the neighboring solutions.

The efficiency and robustness of the memetic algorithm depend on the population. We need distinct local optima.

So, a new local optimum obtained is included into the population if the Hamming distance from this solution to each solution in the population is at least a given threshold. In our computational experiments, we use the threshold $\lceil 0,6p \rceil$.

3.2 Main operators and parameters

The selection, recombination, random modification (mutation), and replacement operators are used in the framework. The well–known tournament selection (Sastry et al. (2005)) is applied to pick up two solutions x^1, x^2. We select k solutions from the population at random and choose the best one as a parent. In our experiments, we put $k = 5$.

The recombination or crossover operator is a variant of the famous uniform crossover (Sastry et al. (2005)). The new solution x will contain all open facilities which are common for the parents. The rest of the open facilities are chosen at random with probability 0,5 from solutions x^1, x^2.

To involve a certain diversification, we use the random modifications of the offsprings. The common bit–flip mutations are not appropriate for the problem, for we may get an unfeasible solution. So, we produce some random modification according to the Swap neighborhood.

To update the population at the Step 2.5, we use the steady–state–no–duplicates technique. We check that no duplicate solutions are added to the population. Moreover, we calculate the Hamming distance between the new solution and the population and update the population if the distance is at least the threshold $\lceil 0,6p \rceil$.

3.3 Tabu search

In the well–known memetic algorithms, the standard local improvement procedure is applied to each element of the population. The algorithm deals with local optima only, and this feature promotes to finding global optimum or near optimal solutions. In our computational experiments for the case $w_j = 1, j \in J$, we discover a lot of local optima and plateaus. The standard local improvement procedure is not efficient for this case (Alekseeva and Orlov (2008)). We need more powerful approach here. To this end, we develop the PTS heuristic and apply it instead of the local improvement. In (Benati and Laporte (1994)) a tabu search algorithm is studied for the problem but a greedy procedure is applied for the follower problem. We use commercial software for it.

To reduce the running time at each step, we use a randomized neighborhood $N_q(x)$, $q > 0$. It is the random part of the Swap neighborhood, where each element is included into the set $N_q(x)$ with probability q independently from other elements. This modification of the basic Tabu Search has the asymptotic convergence property (Goncharov and Kochetov (2002)).

In order to calculate the objective function value for each element of the neighborhood, we need to solve the follower problem. As we have mentioned above, it is an NP-hard problem. To reduce the running time, we replace the problem by its linear programming relaxation. Hence, we have a polynomial time procedure for finding the best element in the neighborhood. Below, we present the general framework of the PTS algorithm.

Probabilistic Tabu Search

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1 Get offspring $x$ from HMA and put $Tabu = \emptyset$.
2 Repeat until the stopping condition is met:
  2.1 Generate the neighborhood $N_q(x)$.
  2.2 If $N_q(x)\backslash Tabu \neq \emptyset$, then find the best element $x'$
      in the set $N_q(x)\backslash Tabu$; else $x' := x$.
  2.3 Put $x := x'$, update $Tabu$.
3 Return the best found solution.
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The set $Tabu$ for the current solution contains some solutions from the Swap neighborhood. The pairs of swapping facilities are storied during the certain number of iterations, and the corresponding solutions are included into the set $Tabu$. We use the PTS algorithm for finding the high quality offspring at Step 2.4 of the HMA framework. Moreover, we collect the high quality solutions for the follower. As we will see below, any family of the follower solutions gives an upper bound for the maximal profit (1) of the leader.

4. UPPER BOUNDS

In (Kochetov et al. (2009)), the main idea of getting an upper bound for the leader profit is to include some additional constraints into the follower problem. The maximal profit for the follower will decrease in this case. So, the total profit for the leader will grow. If the additional constraints allow us to rewrite the bilevel problem as a single level linear 0–1 program, then we may compute the upper bound by CPLEX software. Generating appropriate constraints is the most difficult task here. Below we will illustrate this approach.

Suppose the follower ranks facilities according to some criteria. He uses this ranking instead of the optimal solution of the problem (4)–(8). Without loss of generality, we may assume that this ranking is $1, 2, \ldots, m$. So, the first facility is the most desirable for the follower, while the last facility is the most undesirable one. He opens facilities according to this ranking, taking into account the decision of the leader. If the follower wants to open facility i but it has been opened by the leader, then the follower has to consider facility $i + 1$. Such a strategy does not guarantee the optimality for the follower. So, the leader will get more profit, and we obtain an upper bound.

Now the follower behavior is described by the following constraints. For a given solution x, we have

$$x_i + y_i \geq y_k, \quad i, k \in I, k > i,$$
$$x_i + y_i \leq 1, \quad i \in I,$$
$$\sum_{i \in I} y_i = r.$$

The first constraint forbids to open facility k by the follower if a more preferable facility is available. Two other constraints are taken from the follower problem (4)–(8).

Let us introduce new auxiliary variable $u_{jk} \in \{0, 1\}$ to describe the assignment of the clients to facilities:

$$u_{jk} = \begin{cases} 1 & \text{if client } j \text{ is serviced from facility } k, \\ 0, & \text{otherwise.} \end{cases}$$

Variables u_{jk} are determined from the following constraints by variables x and y:

$$\sum_{k \in I} u_{jk} = 1, \quad j \in J,$$
$$u_{jk} \le x_k + y_k, \quad j \in J, k \in I,$$
$$u_{jk} + \sum_{i \in S_{kj}} u_{ji} \ge x_k + y_k, \quad k \in I, j \in J,$$

where S_{kj} is the set of facilities which are more desirable for client j than the facility k, $S_{kj} = \{i \in I \mid g_{ij} < g_{kj}\}$. According to the first equality, each client must be serviced by the leader or the follower facility. The second constraint guarantees that each client is serviced by an open facility only. The last constraint requires to service each client from the most desirable open facility.

Now variables z_j can be determined from the following constraints by variables u_{jk}, x_k, y_k:

$$z_j \ge u_{jk} - y_k, \quad j \in J, k \in I,$$
$$1 - z_j \ge u_{jk} - x_k, \quad j \in J, k \in I.$$

These constraints require to service client j by the leader facility if $u_{jk} = 1$, $y_k = 0$, and by the follower facility if $u_{jk} = 1$, $y_k = 1$. The optimal values of the variables can be found from the following linear 0–1 program:

$$\max \sum_{j \in J} w_j z_j$$

s.t.

$$\sum_{i \in I} x_i = p,$$
$$\sum_{i \in I} y_i = r,$$
$$y_i + x_i \ge y_k, \quad i, k \in I, k > i,$$
$$\sum_{k \in I} u_{jk} = 1, \quad j \in J,$$
$$u_{jk} \le x_k + y_k, \quad j \in J, k \in I,$$
$$u_{jk} + \sum_{i \in S_{kj}} u_{ji} \ge x_k + y_k, \quad k \in I, j \in J,$$
$$z_j \ge u_{jk} - y_k, \quad j \in J, k \in I,$$
$$1 - z_j \ge u_{jk} - x_k, \quad j \in J, k \in I,$$
$$x_i, y_i, u_{jk}, z_j \in \{0,1\}, \quad k, i \in I, j \in J.$$

The optimal value of the problem provides us the upper bound for the leader profit. We can compute the bound, for example, by CPLEX software.

Another approach to get an upper bound deals with the reduction of opportunities for the follower. We replace the set of all solutions possible for him by a small family $\mathcal{F}$. Clearly, this way gives us an upper bound. Now we rewrite the bilevel problem as a single level problem with a large number of variables and constraints. The number of variables and constraints will depend on the cardinality of family $\mathcal{F}$. If the family is small, we can solve this problem and get a new upper bound.

For $y \in \mathcal{F}$ we put

$$I_j(y) = \bigl\{i \in I | g_{ij} \le \min_{l \in I}(g_{lj} | y_l = 1)\bigr\}, \quad j \in J.$$

This set shows facilities for the leader which allow him to keep client j if the follower will use solution y. Let us introduce new variables:

$$z_{jy} = \begin{cases} 1 & \text{if client } j \text{ is serviced by the leader when} \\ & \text{the follower uses solution } y, \\ 0 & \text{if client } j \text{ is serviced by the follower when} \\ & \text{the follower uses solution } y, \end{cases}$$

$W \ge 0$ is the total profit of the leader.

We need to find a solution x with the maximal profit for the leader when the follower uses family $\mathcal{F}$ for his answer. The correspondent optimization problem can be presented as the linear 0–1 program:

$$\max W$$

s.t.

$$\sum_{j \in J} w_j z_{jy} \ge W, \quad y \in \mathcal{F},$$
$$z_{jy} \le \sum_{i \in I_j(y)} x_i, \quad y \in \mathcal{F}, j \in J,$$
$$\sum_{i \in I} x_i = p,$$
$$x_i, z_{jy} \in \{0,1\}, i \in I, j \in J, y \in \mathcal{F}.$$

The objective function is the total profit of the leader. The first constraint guarantees that the follower uses his best answer from family $\mathcal{F}$. The second constraint determines the market share for each solution of the follower. The last constraint allows the leader to open p facilities only. If we replace family $\mathcal{F}$ by the set of all possible solutions for the follower, we get a reformulation of the bilevel problem (1)–(8). In (Rodriguez and Perez (2008)) a similar reformulation is considered with a lot of additional constraints and variables.

Note that the optimal value for the problem does not change if we replace 0–1 variables z_{jy} by continuous variables $0 \le z_{jy} \le 1$. So, we need to solve the mixed integer linear programming problem with small number of 0–1 variables. Of course, the optimum depends on family $\mathcal{F}$. We create this family by the PTS algorithm at the step 2.4 of the HMA framework.

5. COMPUTATIONAL RESULTS

The developed memetic algorithm is coded in GAMS (General Algebraic Modeling System) and tested on the instances from the benchmark library *Discrete Location Problems* (http://math.nsc.ru/AP/benchmarks/Competitive/p_med_comp_eng.html). For all instances, we have $n = m = 100, p = r = 10, w_j = 1$. The elements of matrix (g_{ij}) are defined as Euclidian distances between points i, j in the two dimensional Euclidean plane. The points are chosen at random with the uniform distribution in the square 7000×7000. All experiments are carried out at the PC Pentium Intel Core 2, 1.87 GHz, RAM 2 Gb, running under the Windows XP Professional operating system. Note that the exact method (Rodriguez and Perez (2008)) allows to solve the instances with $n = m \leq 50$, $r = p \leq 5$ only.

Table 1. Upper and lower bounds

Code	$LB(p)$	$LB(p+r)$	HMA	UB_0	UB_1
111	41	31	50	74	54
211	41	36	49	70	57
311	46	41	48	73	55
411	41	39	49	72	58
511	48	40	48	70	55
611	42	39	47	67	57
711	49	37	51	73	55
811	42	37	48	74	55
911	47	35	49	68	54
1011	46	33	49	70	54

Table 1 shows the upper and lower bounds for the instances. The first column indicates the codes of the instances. Columns $LB(p)$, $LB(p+r)$, HMA present lower bounds for the strategies based on the classical p-median problem, the $(p+r)$-median problem, and the hybrid memetic algorithm. As we can see, the HMA lower bound dominates the other ones. Nevertheless, the differences between HMA and $LB(p)$ bounds are small. For instance 511, these bounds coincide! We observe that $LB(p)$ is a good approximation for the leader behavior. Moreover, the running time for the $LB(p)$ is a few seconds for GAMS(CPLEX) software. However, for HMA bounds, we need about 7 hours of running time if we stop the algorithm after 100 iterations with the population size 25.

Columns UB_0 and UB_1 show two upper bounds from the previous section. To compute UB_0, we need a ranking of the facilities for the follower. Actually, we have $m!$ opportunities for it. It is not clear how to find the best one. In our experiments, we use the ranking obtained by the Lagrangian relaxation approach (Alekseeva and Kochetova (2008)). In order to compute bound UB_1, we need a family $\mathcal{F}$. As we mentioned above, there are a lot of near optimal solutions in these instances. So, we collect some of them by the PTS algorithm at step 2.4 of the memetic framework. We use the families of 400 best found solutions for the follower. The Table 1 shows that the bound UB_1 dominates UB_0 substantially. Nevertheless, we cannot prove the optimality of the HMA lower bounds by increasing the families. We believe that the lower bound corresponds to the global optima for these instances. So, we need more intelligent search strategies for creating family $\mathcal{F}$.

6. CONCLUSIONS AND FURTHER RESEARCH

We consider the competitive p-median problem and develop hybrid memetic algorithm for finding near optimal solutions for this bilevel problem. Commercial software is used to compute optimal strategy for the follower and calculate total profit of the leader. In order to get an upper bound, we introduce some restrictions for the follower behavior and reformulate the bilevel problem as a single level mixed integer programming problem. The optimal solution for this problem provides us the desired upper bound. Our computational results show a gap between the upper and lower bounds. However, it is quite small. We believe that the HMA solutions are optimal, but we need other families to prove the optimality. As further research, it could be interesting to investigate new strategies for the family selection. Some ideas can be taken from the column generation scheme, which can be easily described for the problem. Moreover, it is a promising area for the matheuristics as well.

REFERENCES

E. Alekseeva, N. Kochetova (2008) Upper and lower bounds for the competitive p-median problem. *Proceedings of XIV Baikal International School–Seminar Optimization Methods and Their Applications*, 1, 563–569 (in Russian).

E. Alekseeva, A. Orlov (2008) Memetic algorithm for the competitive p-median problem. *Proceedings of XIV Baikal International School–Seminar Optimization Methods and Their Applications*, 1, 570–576 (in Russian).

S. Benati, G. Laporte (1994) Tabu search algorithms for the $(r|X_p)$–medianoid and $(r|p)$–centroid problems. *Location Science*, 2(4), 193–204.

A. Caprara, P. Toth, M. Fischetti (2000) Algorithms for the Set Covering Problem. *Annals of Operations Research*, 98(1–4), 353–371.

Z. Drenzer, H.A. Eiselt (2004) Consumers in competitve location models. In: Z. Drenzer, H. Hamacher (Eds.) *Facility Location. Applicatios and Theory*, Springer, 151–178.

E. Goncharov and Y. Kochetov (2002) Probabilistic tabu search for the unconstrained discrete optimization problems. *Discrete Analysis and Operations Research*, 9(2), 13–30 (in Russian).

F. Glover, M. Laguna. *Tabu Search.* Kluwer Acad. Publ, Boston, 1997.

S. L. Hakimi (1990) Locations with spatial interactions: competitive locations and games. In: P. B. Mirchandani, R. L. Francis (Eds.) *Discrete Location Theory.* Wiley & Sons. 439–478.

H. Noltemeier, J. Spoerhase, H.-C. Wirth (2007) Multiple voting location and single voting location on trees. *European Journal of Operational Research*, 181, 654–667.

Y. Kochetov, A. Kononov, A. Plyasunov (2009) Competitive facility location models. *Computational Mathematics and Mathematical Physics*, (in press).

N. Mladenović, J. Brimberg, P. Hansen and J. Moreno–Perez (2007) The p-median problem: A survey of metaheuristic approaches. *European Journal of Operational Research*, 179(3), 927–939.

C. M. C. Rodriguez, J. A. M. Perez (2008) Multiple voting location problems. *European Journal of Operational Research*, 191(2), 437–453.

K. Sastry, D. Goldberg, G. Kendall (2005) Genetic algorithms. In: E.K. Burke, G. Kendall (Eds.) *Search Methodologies. Introductory Tutorials in Optimization and Decision Support Techniques.* Springer, 97–126.

I. Vasil'ev, K. Klimentova, Y. Kochetov (2009) New lower bounds for the facility location problem with user preferences. *Computational Mathematics and Mathematical Physics*, (in press).