A matheuristic for the leader-follower facility location and design problem

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Abstract

Two players, a leader and a follower, open facilities and compete to attract clients from a given market. Each player has a budget and maximizes own market share. Each client splits own demand probabilistically over all opened facilities by the gravity rule. The goal is to find the location and design of the leader facilities to maximize his market share. We present a matheuristic for this game based on the best response strategy. Computational results for the discrete games are discussed.

1 Introduction

We consider the discrete facility location and design problem in which two players, a leader and a follower, compete to attract clients from a given market. Each player has a budget and maximizes own market share. Each client splits demand probabilistically over all facilities in the market proportionally with utility to each facility. The location and design of the facilities are to be determined so as to maximize the market share of the leader. For this Stackelberg game we present an alternating heuristic based on the best response strategy. For a given solution of a player, we reformulate the problem for another player as a linear integer program and find the optimal solution by a solver. The algorithm is terminated if we reach a Nash equilibrium or come upon the previously visited solution. Computational experiments indicate that the algorithm takes a small number of steps and produces optimal or near optimal solutions.

2 Mathematical model

Assume that the set I of potential facility locations and the set J of clients are finite. For each facility $i \in I$ we have the set R_i of design scenarios [1] and this set is finite as well. For each pair $i \in I, r \in R_i$ we have the fixed costs f_{ir} and g_{ir} of opening facility i with design scenario r by the leader and by the follower, respectively. Moreover, we know the attractiveness a_{ir} of the leader facility and the similar parameter b_{ir} of the follower facility. The last two features are important for describing the client behavior. Each client j splits own demand w_j probabilistically over all facilities directly proportional with attraction to each facility and inversely proportional to the distance d_{ij} between client j and facility i [1]. We consider the utility function u_{ijr} of leader facility i with design scenario r for client j and the similar function v_{ijr} for follower facility:

$$u_{ijr} = a_{ir}/(d_{ij}+1)^{\beta}, \quad v_{ijr} = b_{ir}/(d_{ij}+1)^{\beta}, \qquad i \in I, r \in R_i, j \in J,$$

where β is a distance sensitivity parameter. Now we introduce the decision variables for the players:
x_{ir} is equal to 1 if facility i is opened by the leader with design scenario r and 0 otherwise;
y_{ir} is equal to 1 if facility i is opened by the follower with design scenario r and 0 otherwise.
For client j, the total utility U_j from the leader facilities and the total utility V_j from the follower facilities are defined as:

$$U_j = \sum_{i\in I}\sum_{r\in R_i} u_{ijr}x_{ir}, \quad V_j = \sum_{i\in I}\sum_{r\in R_i} v_{ijr}y_{ir}, \qquad j \in J.$$

The total market share of the leader is given by $\sum_{j\in J} w_jU_j/(U_j+V_j)$. The leader wishes to maximize own market share, anticipating that the follower will react to the decision by opening own facilities. The market share of the follower is given by $\sum_{j\in J} w_j V_j/(U_j + V_j)$. The follower maximizes own market share. In opposite to [2], we assume that the players can open facilities at the same site. This Stackelberg game can be presented as the following nonlinear 0–1 bilevel optimization problem:

$$\max_x \sum_{j\in J} w_j \frac{\sum_{i\in I}\sum_{r\in R_i} u_{ijr}x_{ir}}{\sum_{i\in I}\sum_{r\in R_i} u_{ijr}x_{ir} + \sum_{i\in I}\sum_{r\in R_i} v_{ijr}y^*_{ir}} \tag{1}$$

subject to

$$\sum_{i\in I}\sum_{r\in R_i} f_{ir}x_{ir} \le B_l; \tag{2}$$

$$\sum_{r\in R_i} x_{ir} \le 1, \qquad i \in I; \tag{3}$$

$$x_{ir} \in \{0,1\}, \qquad r \in R_i, i \in I; \tag{4}$$

where y^*_{ir} is the optimal solution for the follower problem:

$$\max_y \sum_{j\in J} w_j \frac{\sum_{i\in I}\sum_{r\in R_i} v_{ijr}y_{ir}}{\sum_{i\in I}\sum_{r\in R_i} u_{ijr}x_{ir} + \sum_{i\in I}\sum_{r\in R_i} v_{ijr}y_{ir}} \tag{5}$$

subject to

$$\sum_{i\in I}\sum_{r\in R_i} g_{ir}y_{ir} \le B_f; \tag{6}$$

$$\sum_{r\in R_i} y_{ir} \le 1, \qquad i \in I; \tag{7}$$

$$y_{ir} \in \{0,1\}, \qquad r \in R_i, i \in I. \tag{8}$$

Objective functions (1) and (5) are market shares of the players. Inequalities (2) and (6) are the budget constraints: B_l is the budget of the leader, B_f is the budget of the follower. Inequalities (3) and (7) ensure a unique design scenario for each open facility.

3 Matheuristic

Suppose that the leader has made own decision X. To calculate his market share, we need the follower optimal solution. For solving the problem (5)–(8), we introduce a large number W and new variables:

$$z_j = 1/(\sum_{i\in I}\sum_{r\in R_i} v_{ijr}y_{ir} + U_j), \quad y_{ijr} = w_j v_{ijr} z_j, \qquad r \in R_i, i \in I, j \in J.$$

Note that U_j is a constant in the follower problem. We rewrite this nonlinear problem as the following mixed integer linear program:

$$\max \sum_{j\in J}\sum_{i\in I}\sum_{r\in R_i} y_{ijr} \tag{9}$$

subject to (6), (7) and

$$\sum_{i\in I}\sum_{r\in R_i} y_{ijr} + w_j U_j z_j \le w_j, \qquad j \in J; \tag{10}$$

$$0 \le y_{ijr} \le w_j y_{ir}, \qquad r \in R_i, i \in I, j \in J; \tag{11}$$

$$y_{ijr} \le w_j v_{ijr} z_j \le y_{ijr} + W(1 - y_{ir}), \qquad r \in R_i, i \in I, j \in J; \tag{12}$$

$$y_{ir} \in \{0,1\},\ y_{ijr} \ge 0,\ z_j \ge 0, \qquad r \in R_i, i \in I, j \in J. \tag{13}$$

The main idea of the alternating method is the following [4]. For the solution X, we compute the best-possible solution Y for the follower. Once that is done, the leader assumes the role of the follower and reoptimizes his decision by solving the corresponding problem for the given solution Y. This process is then repeated until one of Nash equilibria is discovered or the previously visited solution is detected. The best found solution for the leader is returned as the result of the method. In starting solution, the leader ignores the follower.

$B_l \backslash B_f$	10	20	30	40	50	60	70	80	90	100
10	50	37	28	21	17	**14**	**12**			
20	**63**	50	**39**	**31***	**25**	**22**	**20**			
30	**72**	61	50	**43**	**36**	**31***	**28***	**25**	**23***	
40	**79**	69*	58	50	43*	**39**	**35**	**32***	**30***	
50	**83**	75	64	57*	50	**45**	**41**	**38**	**35**	
60	86	78	68*	61	55	50	**46**	**42**	**40**	38
70	88	80	72*	65	59	54	50	**46**	44	41
80			75	68*	62	58	54	50	47	45
90			77*	70*	65	60	**56**	**53**	50	48
100						**62**	**59**	**55**	**52**	50

Table 1: The leader market share (%), $|J| = |I| = 50$

4 Computational results

We conducted computational experiments to test the method. We consider three design scenarios for each facility: basic $(r = 1)$, average $(r = 2)$, and improved $(r = 3)$ with corresponding attractiveness $a_{i1} = b_{i1} = 3$, $a_{i2} = b_{i2} = a_{i1} + \xi_{i1}$, and $a_{i3} = b_{i3} = a_{i2} + \xi_{i2}$ where ξ_{i1} and ξ_{i2} are chosen at random with uniform distribution from intervals [1,6] and [1,9], respectively. The fixed costs were generated by the rule: $g_{ir} = f_{ir} = \mu_{ir} a_{ir} + \mu'_{ir}$, where μ_{ir} and μ'_{ir} were drawn randomly from the intervals [1,5] and [5,10], respectively. The demand w_j of each client was drawn randomly from the interval [1,10]. The location of each client was generated randomly in the square 100×100. The facilities can be opened at the same sites where the clients are located. The matrix d_{ij} is the Euclidean distance matrix, $\beta = 1$.

Table 1 shows our preliminary results for the different pairs of budgets (B_l, B_f). Note that in case $B_l = B_f$ the leader can get at most half of the market, because the follower can use the same solution as the leader. As we can see, the heuristic gives 50% for the leader. Hence, we get the global optimum. In other cases, we have no upper bound. Following [3], we say that so-called *first entry paradox* occurs if the leader market share is less than $100\% \cdot B_l/(B_l + B_f)$. Numbers in boldface in the Table 1 correspond to the paradox. The average number of steps for the heuristic is about 3, the maximal number is 6 in our experiments. The running time depends on the budgets of the players. For large budgets, the running time can exceed one hour per iteration for PC Pentium Core 2 Duo 2.66 GHz, RAM 2GB. The corresponding cells are empty in the table. In other cases, we find optimal solutions of the problem (5)–(8) in a few minutes. In the most cases we have got a Nash equilibrium. Numbers with stars $(^*)$ indicate the cases where we have no Nash equilibrium. Thus, we may conclude that the method is efficient and, as a rule, the follower has an advantage over the leader in this game, but in average at most 2%.

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