Length-Bounded Maximum Multicommodity Flow with Unit Edge-Lengths

A flow in a digraph $G = (V, E)$ from origin vertex $s \in V$ to destination vertex $t \in V$ is a nonnegative function $f : E \to \mathbf{R}_+$ such that for each node $v \in V,\ v \neq s,\ v \neq t$ holds

$$\sum_{(v',v)\in E} f(v', v) - \sum_{(v,v')\in E} f(v, v') = 0$$

and

$$|f| = \sum_{(s,v')\in E} f(s, v') - \sum_{(v',s)\in E} f(v', s) = \sum_{(v',t)\in E} f(v', t) - \sum_{(t,v')\in E} f(t, v')$$

is the amount of flow sent from s to t in f.

If $P_1, \ldots, P_r$ are paths from s to t, then a sum of path-flows along $P_1, \ldots, P_r$ gives a network flow from s to t again. Given that

$$f = \sum_{j=1}^{r} f^{P_j}$$

, we will say that f *is routed along the set of paths* $P_1, \ldots, P_r$.

We will assume that flow f_i of commodity $i, i = 1, \ldots, k$, has an origin $s_i \in V$ and a destination $t_i \in V$. If $f_1, f_2, \ldots, f_k$ are flows of k commodities, then $F = (f_1, f_2, \ldots, f_k)$ is called a *multicommodity flow* in G.

An input instance of the maximum multicommodity flow (MCF) consists of a directed network $G = (V, E)$, where $|V| = n,\ |E| = m$, an edge capacity function $u : E \to \mathbf{R}_+$ and a specification $(s_i, t_i, d_i) \in V \times V \times \mathbf{R}^+$ of commodity i for $i = 1, \ldots, k$. The objective is to maximize $\sum_{i=1}^{k} |f_i|$, so that the sum of flows on any edge $e \in E$ does not exceed $u(e)$ and $|f_i| \leq d_i,\ i = 1, \ldots, k$.

The Length-Bounded Maximum Multicommodity Flow with Unit Edge-Lengths (LBMCF1) has the same input, extended by an upper bound $L \in \mathbf{Z}^+$ and asks for a maximum MCF where the sum of flows on any $e \in E$ does not exceed $u(e)$, $|f_i| \leq d_i,\ i = 1, \ldots, k$, and the flow of each commodity is routed along a set of paths at most L edges long.

A special case of this problem where all edges have a unit length requires that the flow of each commodity is routed along a set of paths at most L edges long. Obviously, the maximum MCF may be considered a special case of LBMCF1, assuming $L = n$. W.l.o.g. we will assume that all pairs (s_i, t_i) are unique, since otherwise the demands with identical pairs (s_i, t_i) may be summed together in one demand.

An LP formulation of LBMCF1, involving $O(Lkn + m)$ constraints and $O(Lkm)$ variables, may be constructed using a multicommodity flow in a supplementary time-expanded network [2]. The node set V' contains a copy V_t of the node set V of graph G for every discrete time step $t, t = 1, \ldots, L$. For every directed edge $(v,w) \in E$ there is an edge in E' from vertex $v_t \in V_t$ in time layer t to vertex $w_{t+1} \in V_{t+1}$. Besides that, E' contains edges (v_t, v_{t+1}) for all $v_t \in V_t,\ t = 1, \ldots, L-1$. A multicommodity flow is sought in this time-expanded network under additional constraints which require that for each $e = (v,w) \in E$ the sum of all flows traversing the edges $(v_t, w_{t+1}),\ t = 1, \ldots, L-1$ is at most $u(e)$. For all $i = 1, \ldots, k$, the origin of commodity i is placed in the copy s_{i1} of vertex s_i at level 1 and the destination is placed in the copy t_{iL} of vertex t_i at level L. The resulting LP formulation is as follows

$$\max \sum_{i=1}^{k} \sum_{e'=(s_{i1},v_2)\in E'} x_i(e'), \tag{1}$$

$$\sum_{e'=(s_{i1},v_2)\in E'} x_i(e') \leq d_i, \quad i = 1, \ldots, k, \tag{2}$$

$$\sum_{i=1}^{k} \sum_{e'=(v_t,w_{t+1})\in E'} x_i(e') \leq u(e), \quad e = (v,w) \in E, \tag{3}$$

$$\sum_{e'=(v_{t-1},w_t)\in E'} x_i(e') = \sum_{e'=(w_t,v_{t+1})\in E'} x_i(e'), \tag{4}$$

$$i = 1, \ldots, k, \quad w_t \in V_t, \quad t = 2, \ldots, L-1,$$

$$\sum_{e'=(v_{L-1},w_L)\in E'} x_i(e') = 0, \quad i = 1, \ldots, k, \quad w_L \in V_L \backslash \{t_{Li}\}, \tag{5}$$

where variables $x_i(e') \geq 0$ give the amount of flow of commodity i over edge $e' \in E'$.

References

[1] T.C. Hu, *Integer Programming and Network Flows*, Addison-Wesley Publishing Company, Reading, MA, 1970.

[2] P. Kolman and C. Scheideler, *Improved bounds for the unsplittable flow problem*, J. Algor. 61 (1) (2006), pp 20–44.