Multiproduct Scheduling Problem

Let S, I, and U be the sets of products, tasks, and units respectively. It is assumed that each task produces one product and requires one unit. Define the following input data:

- $D_s > 0$ is the demand for product $s \in S$;
- $s_i \in S$ is the product produced by task $i \in I$;
- $u_i \in U$ is the unit suitable for task $i \in I$;
- $r_i > 0$ is the production rate of task $i \in I$, i.e. the amount of product s_i produced in one hour;
- a_{ij} is the duration of the changeover task if unit u is switched from task i to task j (assuming that $u = u_i = u_j$);
- a_i^0 is the duration of the initial changeover task if task i is performed at the first place on unit u_i;
- $T_i^{\min}$ and $T_i^{\max}$ are the lower and upper time limits for one execution of task i.

The problem asks to choose the set of tasks to be performed and to schedule them so that all the demands are fulfilled, and $T_i^{\min}$ and $T_i^{\max}$ time limits are satisfied. As a primary optimization criterion we chose the minimization of the completion time of the last task (Makespan) denoted by $C_{\max}$.

Two secondary criteria are considered: the minimization of the total changeover time and the total processing time on all units. These criteria are added to the objective function with penalty coefficients P_1 and P_2 that are tunable parameters.

The problem is NP-hard [3].

Define the set of positions for each unit u as follows: $K_u := \{1, 2, ..., |I_u|\}$. Introduce the variables:

- $x_{ik} \in \{0, 1\}$ such that $x_{ik} = 1$ if and only if task i is assigned to unit u_i at event point $k \in K_u$.
- $\delta_i \geq 0$ is the duration of task i;
- $\alpha_{uk} \geq 0$ is the duration of a changeover task on unit u between positions k and $(k+1)$;
- $\alpha_u^0 \geq 0$ is the duration of an initial changeover on unit u.

The MILP model is as follows:

$$\text{Min} \;\; C_{\max} + P_1 \sum_{i \in I} \delta_i + P_2 \sum_{u \in U} (\alpha_u^0 + \sum_{k \in K_u} \alpha_{uk}); \tag{1}$$

subject to

$$\sum_{i \in I_u} x_{ik} \leq 1, \quad u \in U,\ k \in K_u; \tag{2}$$

$$\sum_{k \in K_u} x_{ik} \leq 1, \quad u \in U,\ i \in I_u; \tag{3}$$

$$\sum_{i \in I_u} x_{i,k-1} \geq \sum_{i \in I_u} x_{ik}, \quad u \in U,\ k \in K_u, k > 1; \tag{4}$$

$$\sum_{i: s_i = s} r_i \delta_i \geq D_s, \quad s \in S; \tag{5}$$

$$\alpha_{uk} \geq \sum_{i \in I_u} a_{ij} x_{i,k-1} - M(1 - x_{jk}), \quad u \in U,\ j \in I_u,\ k \in K_u,\ k > 1; \tag{6}$$

$$\alpha_u^0 \geq \sum_{i \in I_u} a_i^0 x_{i1}, \quad u \in U; \tag{7}$$

$$\sum_{i \in I_u} \delta_i + \alpha_u^0 + \sum_{k \in K} \alpha_{uk} \leq C_{\max}, \quad u \in U; \tag{8}$$

$$T_i^{\min} \sum_{k \in K} x_{ik} \leq \delta_i \leq T_i^{\max} \sum_{k \in K} x_{ik}, \quad i \in I; \tag{9}$$

$$x_{ik} \in \{0, 1\}, \quad i \in I,\ k \in K; \tag{10}$$

$$\delta_i, \alpha_{uk}, \alpha_u^0 \geq 0, \quad i \in I,\ u \in U,\ s \in S,\ k \in K. \tag{11}$$

Objective function (1) reflects the primary and the secondary criteria. The allocation constraints (2)–(3) state that each event point contains at most one task, and one task is located in at most one event point; (4) ensures continuous usage of event points, i.e. if some event point is occupied on some unit, then the previous event point is occupied as well (this property is useful for modeling the changeover times). Inequalities (5) estimate the produced amount of each product and ensure demands satisfaction. In (6), the changeover time on unit u between event points $k-1$ and k is estimated (here M is a large enough parameter that can be defined as $M = \max_{i,j} a_{ij}$). One can see that the right hand side value equals a_{ij} if and only if $x_{i,k-1} = 1$ and $x_{jk} = 1$, which assures to reserve at least a_{ij} time units for the changeover if tasks i and j are placed consequently in an actual schedule. In (7), the initial changeover times are estimated. Inequalities (8) estimate the total processing and changeover time for each unit and bound it by $C_{\max}$ to be minimized. Inequalities (9) bound the processing time of task i to fit in $[T_i^{\min}, T_i^{\max}]$ if task i is performed, or set $\delta_i = 0$ otherwise.

References

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