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Stability of solutions to linear differential equations of neutral type*

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Abstract. In the present paper we study stability of solutions to systems of linear differential equations of neutral type

$$\frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau), \quad t > \tau,$$

where A, B, D are $n \times n$ numerical matrices, $\tau > 0$ is a delay parameter. Stability conditions of the zero solution to the systems are established, uniform estimates for the solutions on the half-axis $\{t > \tau\}$ are obtained. In the case of asymptotic stability these estimates give the decay rate of the solutions at infinity.

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Introduction

In the present paper we consider systems of linear delay differential equations of the following form

$$\frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau), \quad t > \tau, \tag{1}$$

where A, B, D are $n \times n$ numerical matrices, $\tau > 0$ is a delay parameter. In the case of $D \neq 0$ these equations are called equations of *neutral type*. We study conditions of stability of the zero solution to the systems and obtain uniform estimates for solutions on the half-axis $\{t > \tau\}$. In particular, it follows from these estimates that, in the case of asymptotic stability, for $\|D\| < 1$ all the solutions stabilize $y(t) \to 0$ as $t \to \infty$ with an exponential rate.

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1. The problem of stability of solutions to delay differential equations

For systems of linear differential equations of the form (1) we have the spectral criterion for asymptotic stability (for example, see [1–4]) which is stated in terms of the location of the roots of the quasipolynomial

$$\det(A + e^{-\tau\lambda}B - \lambda I - \lambda e^{-\tau\lambda}D) = 0 \tag{2}$$

in the left half-plane $\mathbf{C}_- = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda < 0\}$. However, the practical verification of this condition is rather difficult because the problem of finding the roots of (2) is ill-conditioned. In the case of $D = B = 0$, there are numerous examples illustrating this fact (for instance, see [5, 6]). Therefore, studying asymptotic stability of solutions to certain systems of the form (1), Lyapunov type theorems are usually taken instead of the spectral criterion (for example, see [1–4]). In this case the main problem is to find a Hermitian matrix $H_0 = H_0^* > 0$ and verify the matrix inequality

$$\begin{pmatrix} H_0A + A^*H_0 + \gamma H_0 & H_0B + A^*H_0D \\ B^*H_0 + D^*H_0A & D^*H_0B + B^*H_0D - \gamma H_0 \end{pmatrix} \leq 0, \quad \gamma > 0,$$

(for example, see [4, chapter 5]). The proof of stability of the zero solution to (1) involves the Lyapunov–Krasovskii functional

$$\begin{aligned} v_0(t,y) &= \langle H_0(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &\quad + \gamma \int\limits_{t-\tau}^{t} \langle H_0 y(s), y(s)\rangle\, ds \end{aligned} \tag{3}$$

over solutions to (1).

It should be noted that the Lyapunov–Krasovskii functional of the form (3) plays also a key role for proving asymptotic stability of solutions. In this respect functional (3) is an analog of the Lyapunov functional $\langle Hy, y\rangle$, where $H = H^* > 0$ is a solution to the matrix equation $HA + A^*H = -I$ for a Hurwitz matrix A. However, the use of functional (3) does not allow us to obtain estimates for the decay rate of the solutions to the differential equations of neutral type (1) at infinity, whereas the Lyapunov functional is used in order to obtain the inequality

$$\|x(t)\| \leq \sqrt{2\|A\|\|H\|}\, e^{-\frac{t}{2\|H\|}} \|x(0)\|, \quad t > 0, \tag{4}$$

for the solutions to the system $\frac{d}{dt}x = Ax$ (see [7, chapter 1]). (Henceforth we consider the spectral norms of matrices.)

Estimates of type (4) play an important role in the study of asymptotic stability of solutions to ordinary differential equations. In particular, they make it possible to estimate the decay rate of the solutions as $t \to \infty$ without finding eigenvalues of A. Therefore, obtaining analogs of such estimates is a very important problem in the theory of delay differential equations.

Observe that behavior of solutions to delay differential equations at infinity in the case of $D = 0$ has been studied in the papers [8, 9]. The authors used the following modification of the Lyapunov–Krasovskii functional

$$V(t, y) = \langle Hy(t), y(t)\rangle + \int\limits_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle \, ds.$$

By the functional, for the system of equations

$$\frac{d}{dt}y(t) = Ay(t) + F(t, y(t), y(t-\tau)), \quad t > \tau,$$

conditions of asymptotic stability of the zero solution and estimates of exponential decay of the solutions at infinity were established.

In the present paper we continue the study of [8, 9]. Using a modification of the Lyapunov–Krasovskii functional, we obtain uniform estimates for norms of solutions to (1) on the whole half-axis $\{t > \tau\}$. In the case of $\|D\| \le 1$ a theorem on stability of the zero solution to (1) follows from these estimates. The obtained estimates give also the decay rate of the solutions as $t \to \infty$ in the case of asymptotic stability.

2. Estimates for the solutions to the equations of neutral type

Consider the initial value problem for (1)

$$\begin{aligned} &\frac{d}{dt}(y(t)+Dy(t-\tau)) = Ay(t)+By(t-\tau), \quad t>\tau, \\ &y(t)=\varphi(t) \quad \text{for} \quad t\in[0,\tau], \end{aligned} \tag{5}$$

where $\varphi(t) \in C^1[0,\tau]$ is a given vector function. It is well known that the initial value problem (5) is uniquely solvable (for example, see [1–3]).

We have the following result.

Theorem 1. *Suppose that there are matrices $H = H^* > 0$, $K(s) = K^*(s) \in C^1[0,\tau]$ such that $K(s) > 0$, $\frac{d}{ds}K(s) < 0$, $s \in [0,\tau]$, and the compound matrix*

$$C = -\begin{pmatrix} HA + A^*H + K(0) & HB + A^*HD \\ B^*H + D^*HA & D^*HB + B^*HD - K(\tau) \end{pmatrix} \tag{6}$$

is positive definite. Let $c_1 > 0$ be the minimal eigenvalue of C, let $k > 0$ be the maximal number such that

$$\frac{d}{ds}K(s) + kK(s) \le 0, \quad s \in [0,\tau]. \tag{7}$$

Then the following inequality holds for a solution to the initial value problem (5)

$$\begin{aligned} &\langle H(y(t)+Dy(t-\tau)), (y(t)+Dy(t-\tau))\rangle + \int\limits_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle ds \\ \le\ & \exp\left(-\frac{\gamma(t-\tau)}{\|H\|}\right)\Big[\langle H(\varphi(\tau)+D\varphi(0)), (\varphi(\tau)+D\varphi(0))\rangle \\ &+ \int\limits_0^{\tau} \langle K(\tau-s)\varphi(s), \varphi(s)\rangle ds\Big], \quad t>\tau, \end{aligned} \tag{8}$$

where

$$\gamma = \min\left\{\frac{c_1}{1+\|D\|^2},\ k\|H\|\right\}. \tag{9}$$

Proof. Let $y(t)$ be a solution to the initial value problem (5). Consider the following modification of the Lyapunov–Krasovskii functional

$$\begin{aligned} V(t,y) &= \langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle \\ &\quad + \int_{t-\tau}^{t} \langle K(t-s)y(s),y(s)\rangle ds. \end{aligned} \tag{10}$$

By positive definiteness of the matrices H and $K(s)$, this functional is positive definite. Differentiating yields

$$\begin{aligned} \frac{d}{dt}V(t,y) &\equiv \left\langle H\left[\frac{d}{dt}(y(t)+Dy(t-\tau))\right],(y(t)+Dy(t-\tau))\right\rangle \\ &\quad + \left\langle H(y(t)+Dy(t-\tau)),\left[\frac{d}{dt}(y(t)+Dy(t-\tau))\right]\right\rangle \\ &\quad + \langle K(0)y(t),y(t)\rangle - \langle K(\tau)y(t-\tau),y(t-\tau)\rangle \\ &\quad + \int_{t-\tau}^{t} \left\langle \frac{d}{dt}K(t-s)y(s),y(s)\right\rangle ds. \end{aligned}$$

Since $y(t)$ is the solution to the equation of neutral type, then

$$\begin{aligned} \frac{d}{dt}V(t,y) &\equiv \langle H(Ay(t)+By(t-\tau)),(y(t)+Dy(t-\tau))\rangle \\ &\quad + \langle H(y(t)+Dy(t-\tau)),(Ay(t)+By(t-\tau))\rangle \\ &\quad + \langle K(0)y(t),y(t)\rangle - \langle K(\tau)y(t-\tau),y(t-\tau)\rangle \\ &\quad + \int_{t-\tau}^{t} \left\langle \frac{d}{dt}K(t-s)y(s),y(s)\right\rangle ds \\ &= \left\langle \begin{pmatrix} HA+A^*H+K(0) & HB+A^*HD \\ B^*H+D^*HA & D^*HB+B^*HD-K(\tau) \end{pmatrix} \times \right. \\ &\quad \left. \times \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix} \right\rangle \\ &\quad + \int_{t-\tau}^{t} \left\langle \frac{d}{dt}K(t-s)y(s),y(s)\right\rangle ds. \end{aligned}$$

Hence, from the definition (6) we have

$$\frac{d}{dt}V(t,y) + \left\langle C\begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}\right\rangle$$

$$-\int\limits_{t-\tau}^{t}\left\langle \frac{d}{dt}K(t-s)y(s), y(s)\right\rangle ds \equiv 0. \tag{11}$$

According to the conditions of the theorem, the matrix C is positive definite and $c_1 > 0$ is its minimal eigenvalue. Therefore,

$$\left\langle C\begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}\right\rangle \geq c_1(\|y(t)\|^2 + \|y(t-\tau)\|^2). \tag{12}$$

Prove the following inequality

$$\begin{array}{rl} & \left\langle C\begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}, \begin{pmatrix} y(t) \\ y(t-\tau)\end{pmatrix}\right\rangle \\ \geq & \dfrac{c_1}{(1+\|D\|^2)\|H\|}\langle H(y(t)+Dy(t-\tau)), (y(t)+Dy(t-\tau))\rangle. \end{array} \tag{13}$$

First show the estimate

$$\|y(t)\|^2 + \|y(t-\tau)\|^2 \geq \frac{1}{1+\|D\|^2}\|y(t)+Dy(t-\tau)\|^2. \tag{14}$$

Indeed, using simple inequalities, we have

$$\begin{array}{rcl} \|y(t)+Dy(t-\tau)\|^2 & \leq & (\|y(t)\| + \|Dy(t-\tau)\|)^2 \\ & \leq & (\|y(t)\| + \|D\|\|y(t-\tau)\|)^2 \\ & = & \|y(t)\|^2 + 2\|y(t)\|\|D\|\|y(t-\tau)\| \\ & & +\|D\|^2\|y(t-\tau)\|^2 \\ & \leq & \|y(t)\|^2 + \|y(t)\|^2\|D\|^2 + \|y(t-\tau)\|^2 \\ & & +\|D\|^2\|y(t-\tau)\|^2 \\ & = & (1+\|D\|^2)(\|y(t)\|^2 + \|y(t-\tau)\|^2). \end{array}$$

Hence, we obtain (14).

By positive definiteness of the matrix H, from (14) we obviously have

$$\begin{array}{rcl} \|y(t)\|^2 + \|y(t-\tau)\|^2 & \geq & \dfrac{1}{(1+\|D\|^2)\|H\|}\langle H(y(t)+Dy(t-\tau)), \\ & & (y(t)+Dy(t-\tau))\rangle. \end{array}$$

It follows from the inequality and (12) that (13).

Using (7) and (13), from (11) we obtain the inequality

$$\frac{d}{dt}V(t,y) + \frac{c_1}{(1+\|D\|^2)\|H\|}\langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle$$
$$+k\int\limits_{t-\tau}^{t}\langle K(t-s)y(s),y(s)\rangle\,ds \le 0.$$

Taking into account (9), we have

$$\frac{d}{dt}V(t,y) + \frac{\gamma}{\|H\|}V(t,y) \le 0, \quad t>0.$$

Hence,

$$V(t,y) \le \exp\left(-\frac{\gamma(t-\tau)}{\|H\|}\right)V(\tau,\varphi), \quad t>\tau.$$

By the definition (10), the inequality coincides with (8).

The theorem is proved. □

Corollary. *Let the conditions of the theorem be satisfied. Then the following estimate holds for a solution to the initial value problem (5)*

$$\|y(t)+Dy(t-\tau)\| \le \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}, \quad t>\tau, \quad (15)$$

where $\gamma > 0$ is defined by (9).

Proof. The proof follows immediately from inequality (8) and the definition (10).

Remark 1. The requirement of positive definiteness of (6) is equivalent to the matrix inequalities

$$D^*HB + B^*HD < K(\tau),$$
$$HA + A^*H + K(0) + (HB + A^*HD)(K(\tau)$$
$$-D^*HB - B^*HD)^{-1}(B^*H + D^*HA) < 0.$$

This fact follows from the well-known theorems on positive definiteness of Hermitian block matrices (for instance, see [10, chapter 7]).

Remark 2. If D is the zero matrix, then (8) and (15) coincide with analogous estimates of [8, 9].

3. Stability of the solutions to the equations of neutral type

The estimates obtained in Section 2 can be used for prooving stability of the zero solution to the equations of neutral type. First, using (15), we establish estimates for a solution to the initial value problem (5).

Theorem 2. *Let the conditions of Theorem 1 be satisfied.*

(a) *If* $\|D\| < q^{-1}$, $q = \exp\left(\frac{\gamma\tau}{2\|H\|}\right)$, *then the inequality holds for a solution to the problem (5)*

$$\|y(t)\| \leq \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)(M(1-q\|D\|)^{-1}+1)\Phi, \quad t > \tau, \tag{16}$$

where

$$\begin{aligned} M &= \sqrt{\|H^{-1}\|(2\|H\|(1+\|D\|^2)+\tau K)}, \\ K &= \max_{s\in[0,\tau]} \|K(s)\|, \\ \Phi &= \max_{s\in[0,\tau]} \|\varphi(s)\|. \end{aligned}$$

(b) *If* $\|D\| = q^{-1}$, *then the inequality holds for a solution to the problem (5)*

$$\|y(t)\| \leq \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\left(M\frac{t}{\tau}+1\right)\Phi, \quad t > \tau. \tag{17}$$

(c) *If* $q^{-1} < \|D\| \leq 1$, *then the inequality holds for a solution to the problem (5)*

$$\|y(t)\| \leq \exp\left(\frac{(t-\tau)}{\tau}\ln\|D\|\right)(Mq(q\|D\|-1)^{-1}+1)\Phi, \quad t > \tau. \tag{18}$$

Proof. Observe that from (15) the inequality follows

$$\|y(t)\| \leq \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} + \|D\|\|y(t-\tau)\|, \quad t > \tau. \tag{19}$$

Then, for $t = \tau + s$, $s \in [0,\tau)$, we have

$$\|y(t)\| \leq \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} + \|D\|\Phi.$$

Using the inequality, from (19) for $t = 2\tau + s$, $s \in [0, \tau)$, we obtain

$$\begin{aligned}\|y(t)\| \quad &\leq \quad \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} \\ &\quad + \exp\left(-\frac{\gamma(t-2\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}\,\|D\| + \|D\|^2\Phi.\end{aligned}$$

Using this estimate, from (19) for $t = 3\tau + s$, $s \in [0, \tau)$, we have

$$\begin{aligned}\|y(t)\| \quad &\leq \quad \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)} \\ &\quad + \exp\left(-\frac{\gamma(t-2\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}\,\|D\| \\ &\quad + \exp\left(-\frac{\gamma(t-3\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}\,\|D\|^2 + \|D\|^3\Phi.\end{aligned}$$

Arguing in similar way, from (19) for $t = l\tau + s$, $s \in [0, \tau)$, we obtain

$$\|y(t)\| \leq \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\sqrt{\|H^{-1}\|V(\tau,\varphi)}\sum_{j=0}^{l-1}(q\|D\|)^j + \|D\|^l\Phi. \tag{20}$$

Using (14) and the definition of $V(\tau, \varphi)$, it is not hard to establish the inequality

$$V(\tau,\varphi) \leq (2\|H\|(1 + \|D\|^2) + \tau K)\Phi^2.$$

By the estimate, from (20) for $t = l\tau + s$, $s \in [0, \tau)$, we obviously have

$$\|y(t)\| \leq \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right) M \sum_{j=0}^{l-1}(q\|D\|)^j + \|D\|^l\right)\Phi. \tag{21}$$

Consider the first case: $\|D\| < q^{-1} = \exp\left(-\dfrac{\gamma\tau}{2\|H\|}\right)$. Represent t in the form $t = l\tau + s$, where $s \in [0, \tau)$. From (21) we have

$$\begin{aligned}\|y(t)\| \quad &\leq \quad \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right) M \sum_{j=0}^{\infty}(q\|D\|)^j + \exp\left(-\frac{\gamma l\tau}{2\|H\|}\right)\right)\Phi \\ &\leq \quad \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right) M(1 - q\|D\|)^{-1} + \exp\left(-\frac{\gamma(t-\tau)}{2\|H\|}\right)\right)\Phi;\end{aligned}$$

i. e., (16) is proved.

Consider the second case: $\|D\| = q^{-1}$. Let $t = l\tau + s$, where $s \in [0, \tau)$. Obviously, from (21) we obtain

$$\|y(t)\| \le \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|} \right) Ml + \exp\left(-\frac{\gamma l \tau}{2\|H\|} \right) \right) \Phi.$$

This inequality yields immediately (17).

Consider the third case: $q^{-1} < \|D\| \le 1$. Choosing $t = l\tau + s$, $s \in [0, \tau)$, from (21) we have

$$\begin{aligned} \|y(t)\| &\le \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|} \right) M (q\|D\|)^{l-1} \sum_{j=0}^{\infty} (q\|D\|)^{-j} + \|D\|^l \right) \Phi \\ &= \left(\exp\left(-\frac{\gamma(t-\tau)}{2\|H\|} \right) M q^l (q\|D\| - 1)^{-1} + 1 \right) \|D\|^l \Phi. \end{aligned}$$

Taking into account the inequalities $0 \le s < \tau$, we obtain

$$\begin{aligned} \|y(t)\| &\le \exp\left(l \ln \|D\| \right) \left(Mq(q\|D\| - 1)^{-1} + 1 \right) \Phi \\ &\le \exp\left(\frac{(t-\tau)}{\tau} \ln \|D\| \right) \left(Mq(q\|D\| - 1)^{-1} + 1 \right) \Phi; \end{aligned}$$

i. e., (18) is proved.

The theorem is proved. □

The next theorem on stability follows from Theorem 2.

Theorem 3. *Let the conditions of Theorem 1 be satisfied. If $\|D\| < 1$, then the zero solution to (2) is asymptotically stable. If $\|D\| = 1$, then the zero solution to (2) is stable.*

Proof. If $\|D\| < 1$, then asymptotic stability of the zero solution follows from (16)–(18). If $\|D\| = 1$, then stability of the zero solution obviously follows from (18).

The theorem is proved. □

Remark 3. By (16)–(18), the rate of the convergence

$$y(t) \to 0, \quad t \to \infty,$$

is exponential for $\|D\| < 1$.

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