A Hybrid Exact Method for the Discrete ($r|p$)–centroid Problem

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Outline

1. Linear 0-1 bi-level formulation of the discrete (*r*|*p*)-centroid problem
2. Reformulation and upper bounds
3. Non-classical column generation method
4. A new matheuristic: Hybrid Iterative Exact Method
5. Computation results
6. Conclusions

Discrete (*r*|*p*)–centroid Problem

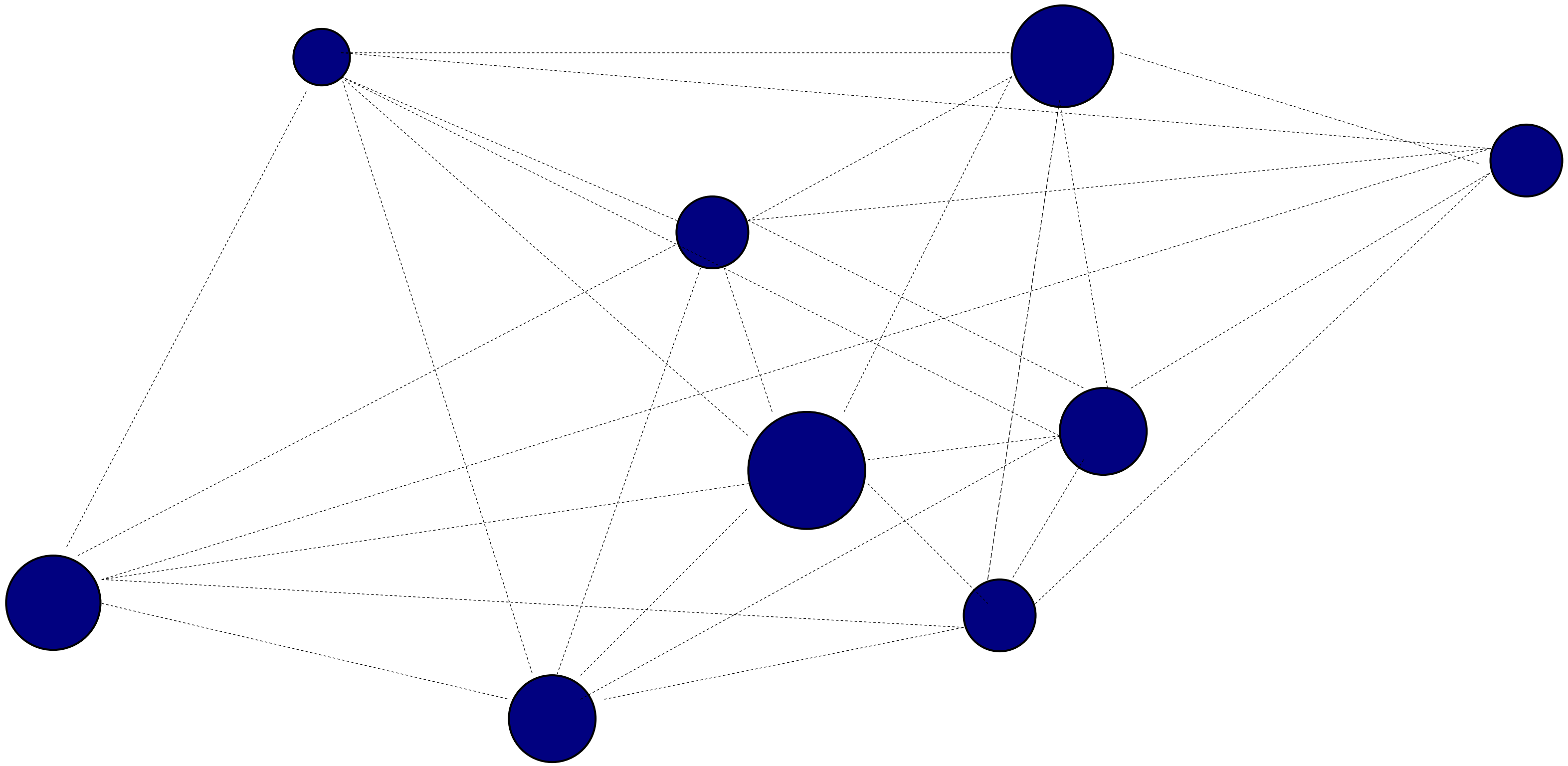
- Input:

J	the set of users
I	the set of potential facilities
p	the total number of facilities opened by the **Leader**
r	the total number of facilities opened by the **Follower**
w_j	the profit received from the user *j*
g_{ij}	the distance between the user *j* and the facility *i*

- Output: *p* facilities opened by the **Leader**;

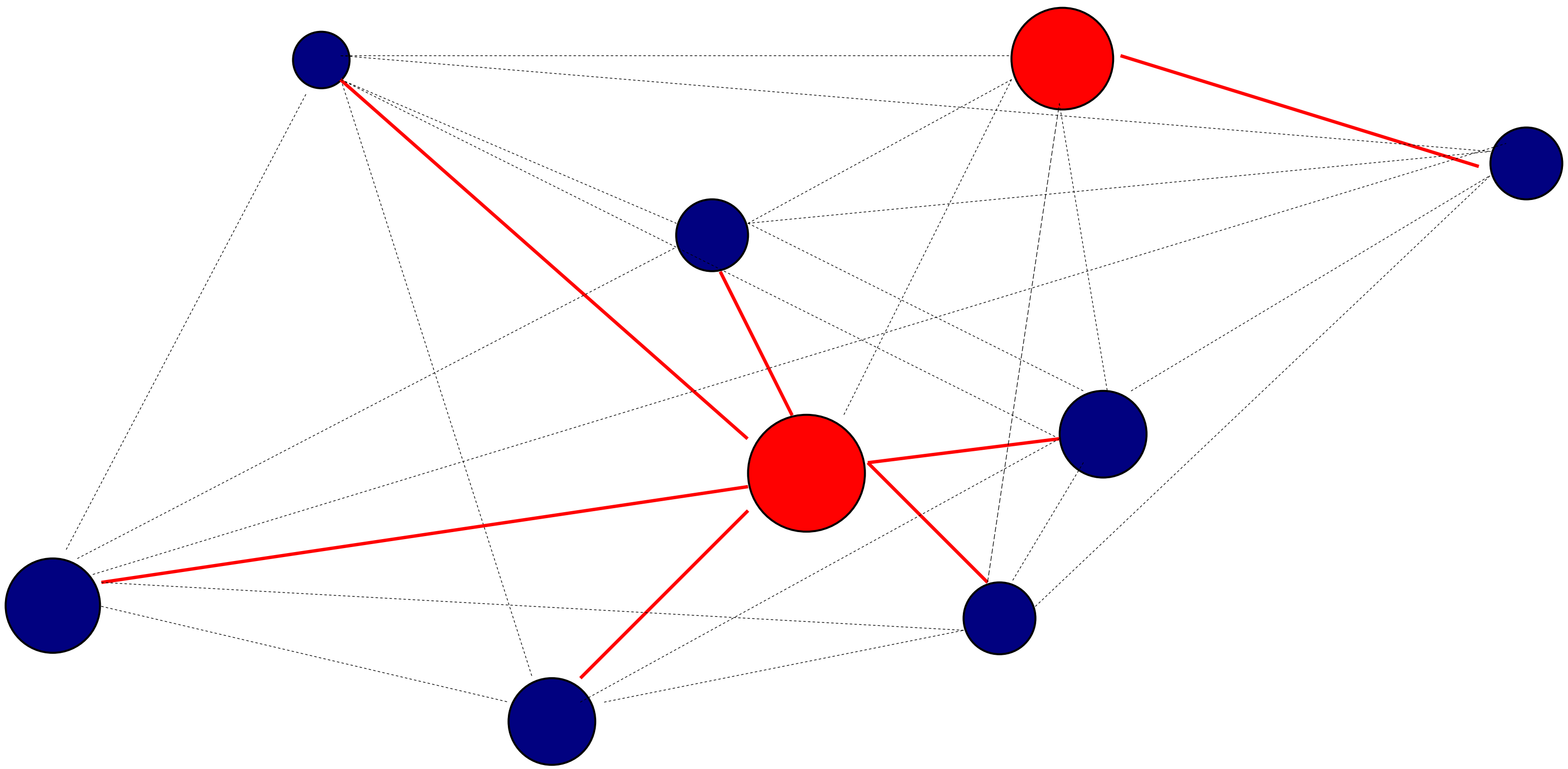
- Goal: **maximize the total profit for the Leader**.

Example



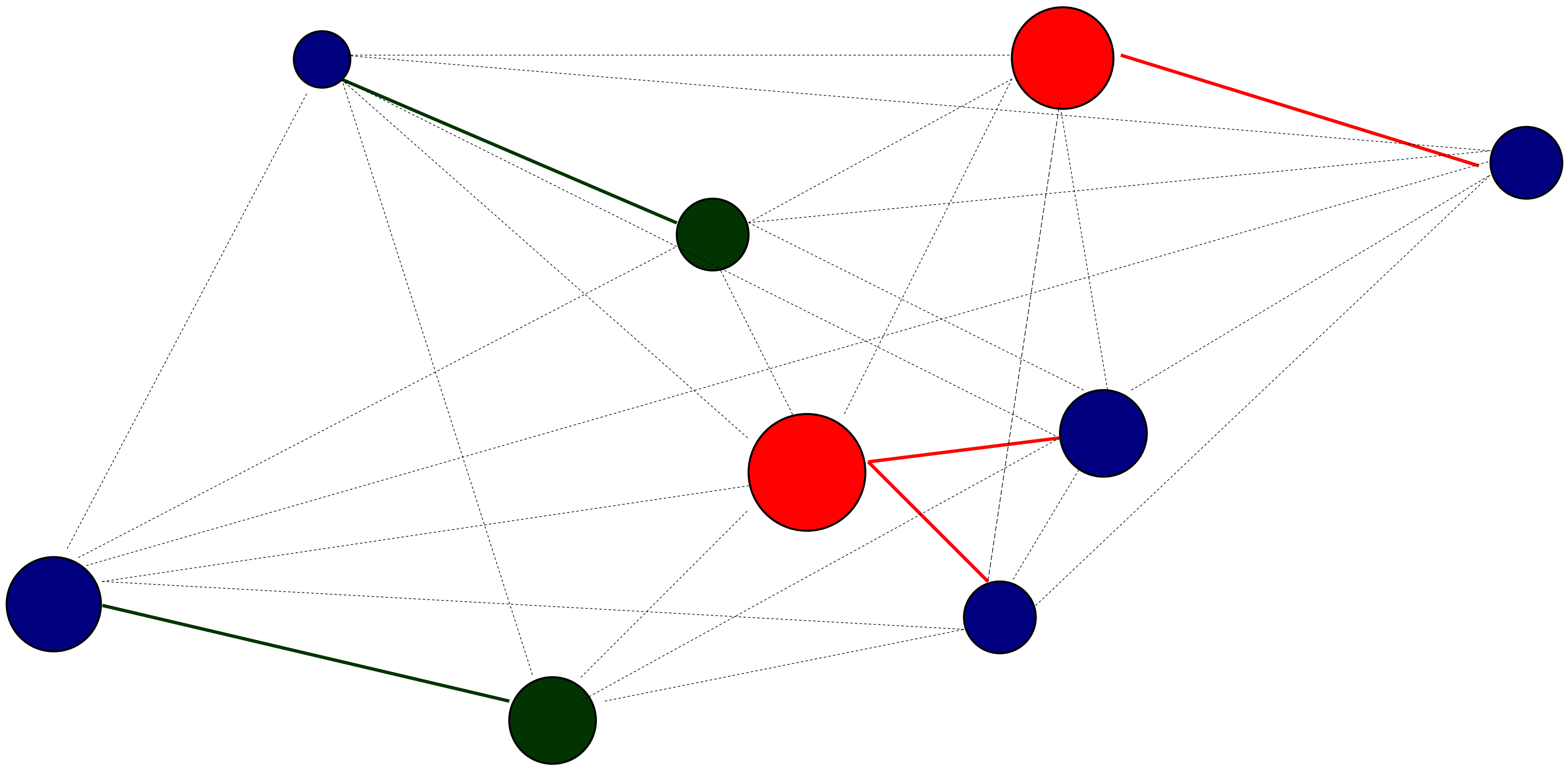
$I=J,\ |I|=9$

Leader has opened p facilities.
Leader's market share is 100%.



$I=J$, $|I|=9$, $p=2$

Follower has opened *r* facilities.
Leader's market share is 56%.



I=*J*, |*I*|=9, *p*= *r*=2

Mathematical Formulation

Leader Variables
$$x_i = \begin{cases} 1, & \text{if the Leader opens facility } i, \\ 0, & \text{otherwise,} \end{cases}$$

Follower Variables
$$y_i = \begin{cases} 1, & \text{if the Follower opens facility } i, \\ 0, & \text{otherwise,} \end{cases}$$

User Variables
$$u_j = \begin{cases} 1, & \text{if user } j \text{ is serviced by the Leader,} \\ 0, & \text{if user } j \text{ is serviced by the Follower.} \end{cases}$$

For the given solution $x_i,\ i \in I$ we define the set of facilities

$$I_j(x) = \{ i \in I \mid g_{ij} < \min_{k \in I}(g_{kj} \mid x_k = 1) \}$$

which allows to "capture" the user j by the Follower.

Bilevel 0-1 Model

$$\max_{x} \sum_{j \in J} w_j u_j^*(x, y^*)$$

$$\text{s.t.} \quad \sum_{i \in I} x_i = p, \quad x_i \in \{0,1\}, \; i \in I$$

where $u_j^*(x, y^*), y_i^*$ is the optimal solution of the Follower problem:

$$\max_{u_j, y_i} \sum_{j \in J} w_j (1 - u_j)$$

$$\text{s.t.} \quad 1 - u_j \le \sum_{i \in I_j(x)} y_i, \quad j \in J$$

$$\sum_{i \in I} y_i = r$$

$$y_i, u_j \in \{0,1\}, \; i \in I, \; j \in J$$

Complexity Status

<table>
<tr><td rowspan="3">(r|p)-centroid</td><td colspan="2">$\sum_2^P$-hard on graph,
H.Noltemeier,J.Spoerhase, H.Wirth,2007</td></tr>
<tr><td>NP-hard on spider</td><td rowspan="4">J.Spoerhase,
H.-C.Wirth,2008</td></tr>
<tr><td>$O(pn^4)$ on path</td></tr>
<tr><td rowspan="2">(1|p)-centroid</td><td>$O(n^2(\log n)^2 \log W)$ on tree, where $W = \sum_{j \in J} w_j$</td></tr>
<tr><td>NP-hard on pathwidth bounded graph</td></tr>
<tr><td>(1|1)-centroid</td><td colspan="2">$O(n^3)$ C. M. Campos Rodríguez and
J. A. Moreno Peréz, 2003</td></tr>
</table>

Computational Methods

- ✓ *Tabu search algorithm*, $|I|=|J|=70, p, r \leq 3$

S. Benati, G. Laporte, 1994

- ✓ *An alternating heuristic on the plane*, $|J| \leq 100, p, r \leq 25$

J. Bhadury, H. A. Eiselt, J. H. Jaramillo, 2001

- ✓ *The partial enumeration algorithm*, $|I| \leq 50, |J| \leq 100, \ p, r \leq 5$

C.M.C. Rodríguez, J.A.Moreno Pérez, 2008

- ✓ *Hybrid memetic algorithm,* $|I|=|J|=100, p=r \leq 10$

E. Alekseeva, N. Kochetova, Y. Kochetov, A. Plyasunov, 2009

Reformulation of the problem

Let F be the set of all feasible solutions of the Follower.

For $y \in F$ define $I_j(y) = \{i \in I \mid g_{ij} < \min_{k \in I}(g_{kj} \mid y_k = 1)\},\ j \in J$

the set of the Leader's facilities which allows the Leader to keep client *j* if the Follower will use the solution y.

Introduce new variables:

$$z_{jy} = \begin{cases} 1, \textit{if user } j \textit{ is serviced by the Leader when the Follower} \\ \textit{uses solution } y \\ 0, \textit{if user } j \textit{ is serviced by the Follower when the Follower} \\ \textit{uses solution } y \end{cases}$$

Large-Scale Integer Linear Program

if F contains all possible Follower's solutions then the initial problem is equivalent to the following problem:

$$\max_{W,\, x,\, u} W$$

s.t.

$$\sum_{i \in I} x_i = p$$

$$\sum_{j \in J} w_j z_{jy} \geq W, \quad y \in F$$

$$z_{jy} \leq \sum_{i \in I_j(y)} x_i, \quad j \in J, \quad y \in F$$

$$z_{jy} \in \{0, 1\}, \quad j \in J, \quad y \in F$$

$$x_i \in \{0,1\}, \quad i \in I$$

Non-classical Iterative Column Generation Method

1. Choose an initial family F

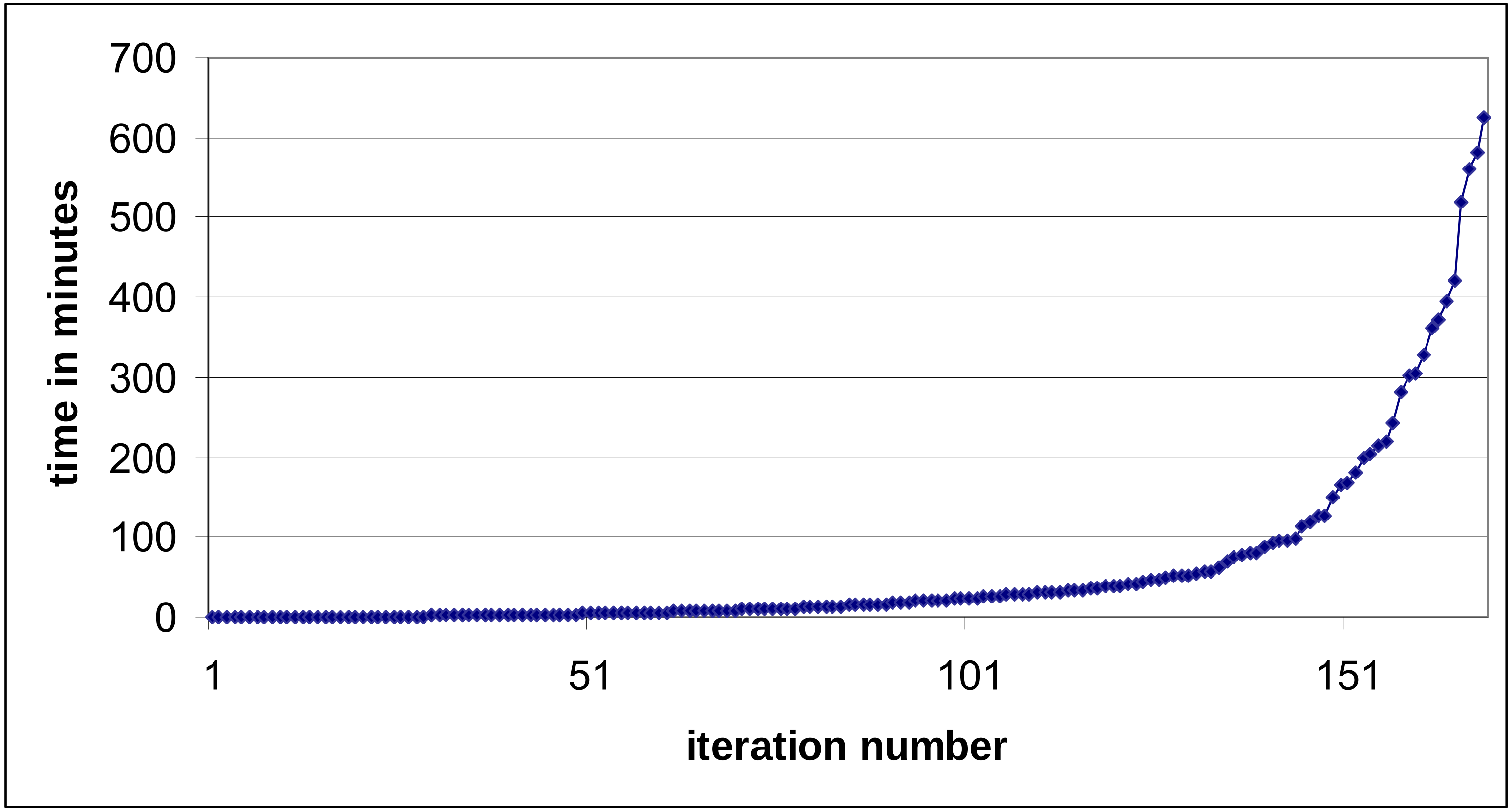
2. Find an upper bound $W(F)$ and a solution $x(F)$

3. Solve the Follower problem find $y(F)$ and calculate $LB(F)$

4. If $W(F) = LB(F)$ then return the best found solution and stop

5. Add $y(F)$ in the family F go to the step 2.

CPLEX Running Time per Iteration



Feasibility Problem

Let F be the subset of a set of all feasible solutions of the Follower,
W^* – the optimum for the initial bi-level problem

$$\sum_{j \in J} w_j z_{jy} > W^*, \; y \in F$$

$$z_{jy} \le \sum_{i \in I_j(y)} x_i, \; j \in J, \; y \in F$$

$$\sum_{i \in I} x_i = p$$

$$0 \le z_{jy} \le 1, \; j \in J, \; y \in F$$

$$x_i \in \{0, 1\}, \; i \in I$$

Hybrid Iterative Exact Method

1 Find the best value W^* by Hybrid Memetic Algorithm

2 Generate an initial family F

3 Find the *batch of the best leader's solutions* by Probabilistic Tabu Search

4 Update the family

5 Solve the feasibility problem exactly

6 If the feasibility problem is infeasible then ***stop***,
else update *the family* and go to step 3

Computational Experiments

http://math.nsc.ru/AP/benchmarks/english.html

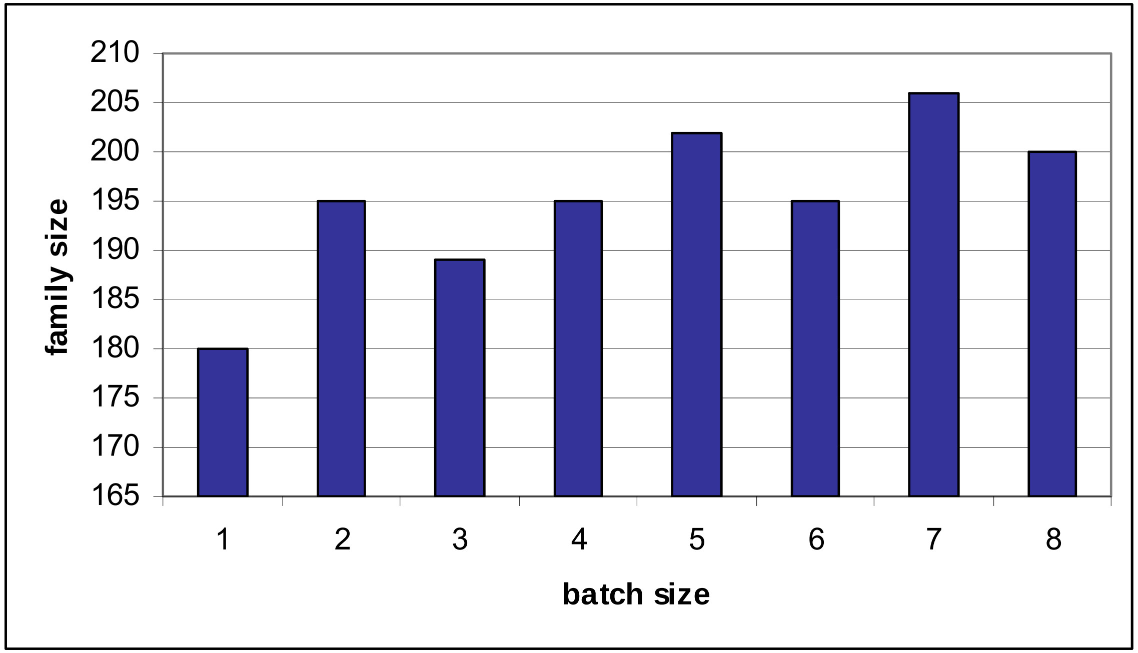
The sets $I = J$, $|I|=|J|$

The element g_{ij} is an Euclidean distance between points $i \in I$ and $j \in J$, the points are randomly generated following the uniform distribution on a 7000*7000 square

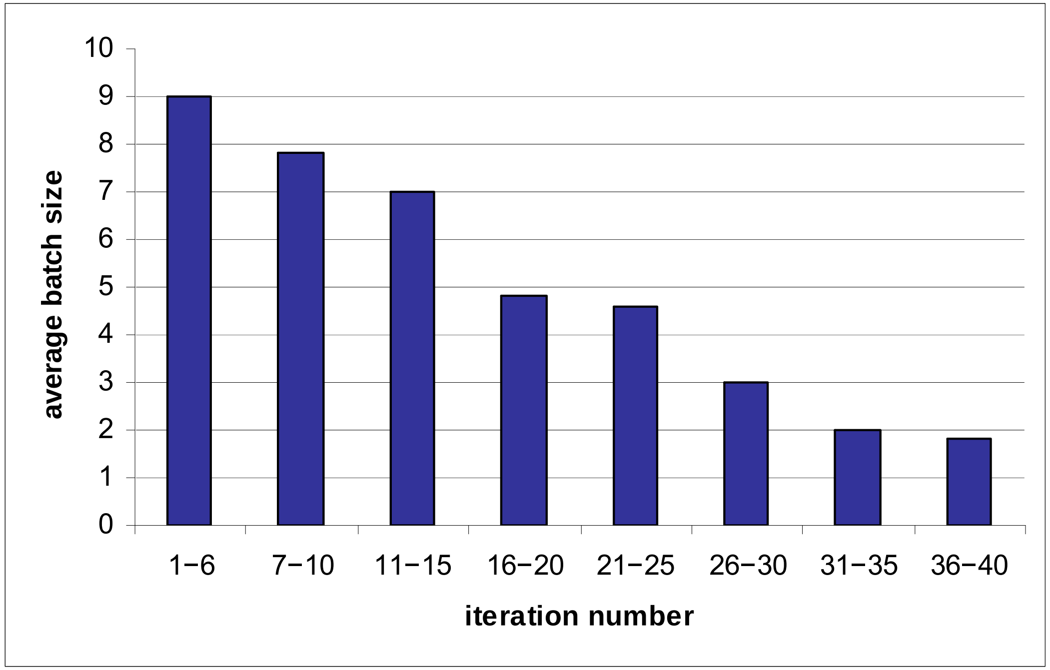
The profit w_j, $j \in J$ is generated randomly following the uniform distribution on a $(0,200)$ interval

PC Pentium Intel Core 2, 1.87 GHz, RAM 2Gb, Windows XP Professional operating system, CPLEX

Parameters for Hybrid Exact Method.
Correlation between Batch Size and Family Size



Parameters for Hybrid Exact Method: Batch Size



Computational results. Optimal solutions
|*I*|=|*J*|=100, *p*=*r*=5

$w_j = 1,\ j \in J$		
opt	|*F*|	time (min)
47	123	120
48	69	60
45	231	3600
47	111	150
47	106	120
47	102	90
47	115	180
48	67	42
47	108	160
47	124	165

$w_j \in (0,200),\ j \in J$		
opt	|*F*|	time (min)
4139	98	65
4822	127	37
4215	262	5460
4678	128	900
4594	190	720
4483	121	660
5153	167	2550
4404	190	720
4700	247	2520
4923	83	30

Computational results

$|I|=|J|=100$, $p=r=10$, $w_j \in (0,1)$, $j \in J$

Code	Lower Bound	Upper Bound	Family size	Time for CPLEX (minutes)	Time for Tabu Search (minutes)
111	4361	5036 (10%)	105	83	58
211	5310	5575 (5%)	210	343	90
311	4483	4931 (10%)	121	188	85
411	4985	5234 (5%)	217	402	95
511	4876	5363 (10%)	171	216	117
611	4587	5045 (10%)	168	160	140
711	5463	6009 (10%)	118	150	63
811	4537	4990 (10%)	161	320	140
911	5302	5567 (5%)	142	175	91
1011	4936	5429 (10%)	160	180	120

Conclusions

- ✓ $\sum_2^P$-hard problem has been studied
- ✓ A new MIP reformulation with the exp number of constraints has been suggested
- ✓ A new hybrid exact method based on a non-classical column generation approach has been proposed
- ✓ Solutions with at most 10% gap for the instances with $|I|=|J|=100$ and $p=r=10$ have been found