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Approximate Algorithms for the Competitive Facility Location Problem

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Abstract—We consider the competitive facility location problem in which two competing sides (the Leader and the Follower) open in succession their facilities, and each consumer chooses one of the open facilities basing on its own preferences. The problem amounts to choosing the Leader's facility locations so that to obtain maximal profit taking into account the subsequent facility location by the Follower who also aims to obtain maximal profit. We state the problem as a two-level integer programming problem. A method is proposed for calculating an upper bound for the maximal profit of the Leader. The corresponding algorithm amounts to constructing the classical maximum facility location problem and finding an optimal solution to it. Simultaneously with calculating an upper bound we construct an initial approximate solution to the competitive facility location problem. We propose some local search algorithms for improving the initial approximate solutions. We include the results of some simulations with the proposed algorithms, which enable us to estimate the precision of the resulting approximate solutions and give a comparative estimate for the quality of the algorithms under consideration for constructing the approximate solutions to the problem.

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INTRODUCTION

Under study is a problem generalizing the well-known maximum facility location problem [1, 6]. In this model, in contrast to the classical facility location problem, we consider two competing sides (the Leader and the Follower) opening their facilities with the goal of maximizing profit. Meanwhile, each of the consumers, basing on their own preferences, chooses the best among all open facilities and, therefore, brings profit to either side. By analogy with the Stackelberg game [13], we represent the decision-making process in this model as consisting of the three stages: at the first stage, the Leader opens its facilities; at the second stage, the Follower, knowing the Leader's facility locations, opens its facilities; and finally, at the third stage, each consumer possessing information about all open facilities chooses the best one.

Nowadays the literature can be considered vast in which this process of competitive facility location is formalized using mathematical programming problems (see [3, 5, 7, 8, 10–12]). These articles pay much attention to discussing various concepts of optimal solutions. In addition, the majority of models under consideration use constraints in the form of equalities on the number of open facilities. In this regard we should mention [7], where some fixed opening costs are introduced as in the classical facility location problem.

We state the competitive facility location problem under study as a two-level mathematical programming problem [4]. This problem is considered in [2] which, in the linear case, includes an equivalent statement as a two-level pseudo-Boolean programming problem and proposes a method for calculating an upper bound for the values of the objective function.

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In this article, we refine the concept of an optimal solution to the competitive facility location problem and consider the so-called optimal noncooperative solutions. Developing the results of [2], we propose a method for calculating an upper bound for the optimal value of the objective function. The algorithm amounts to constructing a classical facility location problem of a particular kind and finding the optimal value of its objective function. The range of applicability of this method is wider than that of the linear competitive facility location problem. While using it, we assume that, for every consumer, the profit obtained by the facilities is a nonincreasing quantity on the set of facilities that are put into the decreasing order of the given consumer's preferences.

In addition, we propose some algorithms for constructing approximate solutions to the problem that amount to a local search procedure [9] with respect to a neighborhood of a particular kind. The local search starts from an initial approximate solution obtained simultaneously with the calculation of an upper bound, while the neighborhood used in the algorithms is constructed so that it includes a comparatively small number of promising variants for adjusting the current solution.

In Section 1, we formulate the competitive facility location problem as a two-level integer programming problem and introduce the concept of an optimal noncooperative solution. In Section 2, we propose a method for constructing an upper bound for the objective function of the optimal noncooperative solution problem and present an algorithm for calculating the upper bound. Section 3 describes the algorithms for constructing approximate solutions, which amount to the local search procedures with a neighborhood of a particular kind and differ in the methods for choosing an element of the neighborhood which improves the current solution. We include the results of simulations with these algorithms, which enables us to estimate the precision of the resulting approximate solutions and make some comparative analysis of the quality of the proposed algorithms.

1. STATEMENT OF THE PROBLEM

Let $I = \{1, \dots, m\}$ denote the set of possible sites for locating the facilities, and $J = \{1, \dots, n\}$, the set of consumers. Assume that, for every $i \in I$, we are given the quantities f_i and g_i equal respectively to the fixed expenses of the Leader and the Follower on opening facility i. Given $i \in I$ and $j \in J$, let p_{ij} denote the profit made at facility i by serving consumer j.

Assume that, for every $j \in J$, the set I is endowed with some order $\succ_j$ indicating the preferences of consumer j in choosing a facility. The relationship $i \succ_j k$ for $i, k \in I$ means that out of the two open facilities i and k consumer j chooses facility i. Assume also that the relationship $i \succcurlyeq_j k$ for $i, k \in I$ means that either $i \succ_j k$ or $i = k$.

In order to express the problem formally, introduce the following variables:

$$x_i = \begin{cases} 1, & \text{if the Leader opens facility } i \in I, \\ 0, & \text{otherwise,} \end{cases}$$

$$x_{ij} = \begin{cases} 1, & \text{if facility } i \in I \text{ is the best for consumer } j \in J \\ & \text{among all facilities opened by the Leader,} \\ 0, & \text{otherwise,} \end{cases}$$

$$z_i = \begin{cases} 1, & \text{if the Follower opens facility } i \in I, \\ 0, & \text{otherwise,} \end{cases}$$

$$z_{ij} = \begin{cases} 1, & \text{if facility } i \in I \text{ is the best for consumer } j \in J \\ & \text{among all facilities opened by the Leader and the Follower,} \\ 0, & \text{otherwise.} \end{cases}$$

Now consider the competitive facility location problem as

$$\max_{(x_i),(x_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J}\Big(\sum_{i\in I} p_{ij}x_{ij}\Big)\Big(1-\sum_{i\in I}\widetilde{z}_{ij}\Big)\Big\}, \quad (1)$$

$$x_i + \sum_{k|i\succ_j k} x_{kj} \le 1, \qquad i\in I, \quad j\in J, \quad (2)$$

$$x_i \ge x_{ij}, \qquad i\in I, \quad j\in J, \quad (3)$$

$$x_i, x_{ij} \in \{0,1\}, \qquad i\in I, \quad j\in J, \quad (4)$$

$((\widetilde{z}_i),(\widetilde{z}_{ij}))$ is the optimal solution to the problem (5)

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{i\in I}\sum_{j\in J} p_{ij}z_{ij}\Big\}, \quad (6)$$

$$x_i + z_i + \sum_{k|i\succ_j k} z_{kj} \le 1, \qquad i\in I, \quad j\in J, \quad (7)$$

$$z_i \ge z_{ij}, \qquad i\in I, \quad j\in J, \quad (8)$$

$$z_i, z_{ij} \in \{0,1\}, \qquad i\in I, \quad j\in J. \quad (9)$$

The objective function (1) of this problem expresses the profit of the Leader taking into account the loss of some consumers captured by the Follower. The inequality in (2) corresponds to the rule for a consumer choosing the most preferable facility among all facilities opened by the Leader. It guarantees that every consumer can choose to be served by at most one open facility. The constraint (3) means that the consumer can choose to be served only at an open facility. The objective function and the constraints of the problem (6)–(9) have similar meanings. The objective function (6) expresses the total profit of the Follower, while (7) guarantees that the consumer's choice rule is fulfilled. Apart from that, the constraint in (7) shows that if a facility is opened by the Leader then it cannot be opened by the Follower.

This is a two-level mathematical programming problem. As every problem of this kind, it includes the upper level problem (1)–(4), which we call the *Leader's problem* and denote by L, and the lower level problem (6)–(9), which we call the *Follower's problem* and denote by F. We denote (1)–(9) by (L,F).

Let $X=((x_i),(x_{ij}))$ denote an admissible solution to Problem L, while $Z=((z_i),(z_{ij}))$ and $\widetilde{Z}=((\widetilde{z}_i),(\widetilde{z}_{ij}))$, respectively admissible and optimal solutions to Problem F for a fixed admissible solution X. Refer to the pair $(X,\widetilde{Z})$ as an *admissible solution* to Problem (L,F).

Let $L(X,\widetilde{Z})$ be the value of the objective function (1) at an admissible solution $(X,\widetilde{Z})$, while $F(Z)$, the value of the objective function (6) at an admissible solution Z. If we assume that, for every admissible solution X, all admissible solutions $(X,\widetilde{Z}_1)$ and $(X,\widetilde{Z}_2)$ satisfy $L(X,\widetilde{Z}_1)=L(X,\widetilde{Z}_2)$ then the optimal solution to (L,F) is an admissible solution $(X^*,\widetilde{Z}^*)$ with $L(X^*,\widetilde{Z}^*)\ge L(X,\widetilde{Z})$ for every admissible solution $(X,\widetilde{Z})$.

If this condition is violated then, in order to determine the optimal solution to Problem (L,F), we need additional information about the optimal solution to Problem F chosen to calculate the objective function of (L,F). In other words, it is necessary to make more precise the goal of the Follower in choosing the locations of its facilities. Depending on that, we can interpret the concept of an optimal solution to Problem (L,F) differently.

Assume henceforth that the Follower, while choosing its best solutions, obeys the so-called *noncooperative behavior* rule, when out of all optimal solutions to F it chooses that with the smallest value of the objective function of (L,F).

Given a fixed admissible solution X to Problem L, refer to an optimal solution $\overline{Z}$ to F as an *optimal noncooperative solution* if every optimal solution $\widetilde{Z}$ to F satisfies $L(X,\overline{Z})\le L(X,\widetilde{Z})$. Refer to an admissible solution $(X,\overline{Z})$ to Problem (L,F) as an *admissible noncooperative solution* if $\overline{Z}$ is an optimal noncooperative solution to F. Refer to an admissible noncooperative solution $(X^*,\overline{Z}^*)$ to

Problem (L, F) as an *optimal noncooperative solution* if every admissible noncooperative solution $(X, \overline{Z})$ to (L, F) satisfies $L(X^*, \overline{Z}^*) \geq L(X, \overline{Z})$. Meanwhile, refer to $L(X^*, \overline{Z}^*)$ as the *optimal* value of the objective function of Problem (L, F).

Refer to an admissible noncooperative solution $(X, \overline{Z})$ to (L, F) also as an *approximate* solution to Problem (L, F). It is clear that we can construct an approximate solution $(X, \overline{Z})$ to (L, F) from an admissible solution X to L. The corresponding optimal noncooperative solution $\overline{Z}$ to Problem F is found using a two-stage algorithm.

At Stage 1, for a fixed solution X, we solve problem F and calculate the optimal value $F(\widetilde{Z})$ of its objective function.

At Stage 2, for a fixed solution X, we solve the following auxiliary problem:

$$\min_{(z_i),(z_{ij})} \sum_{j \in J} \left(\sum_{i \in I} p_{ij} x_{ij} \right) \left(1 - \sum_{i \in I} z_{ij} \right), \tag{10}$$

$$x_i + z_i \sum_{k \mid i \succ_j k} z_{kj} \leq 1, \qquad i \in I, \quad j \in J, \tag{11}$$

$$z_i \geq z_{ij}, \qquad i \in I, \quad j \in J, \tag{12}$$

$$-\sum_{i \in I} g_i z_i + \sum_{i \in I} \sum_{j \in J} p_{ij} z_{ij} \geq F(\widetilde{Z}), \tag{13}$$

$$z_i, z_{ij} \in \{0, 1\}, \qquad i \in I, \quad j \in J. \tag{14}$$

The optimal solution $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$ to this problem is the required optimal solution to Problem F.

2. UPPER BOUND FOR THE OPTIMAL VALUE OF THE OBJECTIVE FUNCTION OF PROBLEM (L, F)

In order to estimate the proximity of approximate and optimal noncooperative solutions to Problem (L, F), we construct an algorithm for calculating an upper bound for the optimal value of the objective function of (L, F). When constructing an upper bound, we assume additionally that, for every $j \in J$, the quantities p_{ij}, $i \in I$, are *nonincreasing* with respect to the order $\succ_j$: for all $i, k \in I$ with $i \succ_j k$, we have $p_{ij} \geq p_{kj}$. For instance, the quantities p_{ij}, $i \in I$ and $j \in J$, defined as follows enjoy this property. Suppose that, for every$j \in J$, we are given an element $i_j \in I$ determining the set $A_j = \{i \in I \mid i \succcurlyeq_j i_j\}$ of admissible facilities for serving consumer j. Suppose that, for every $i \in A_j$, the profit at facility i by serving consumer j is independent of the facility and is equal to b_j. In this situation, for every $j \in J$, the quantities p_{ij}, $i \in I$, are determined as

$$p_{ij} = \begin{cases} b_j, & \text{if } i \in A_j, \\ 0 & \text{otherwise,} \end{cases}$$

and enjoy the indicated property.

For every $j \in J$, define certain sets $I_j \subset I$ used to construct the required upper bound. To this end, for fixed $j_0 \in J$, we state some conditions enabling us to decide for every $i \in I$ whether $i \in I_{j_0}$ or $i \notin I_{j_0}$.

Given $i \in I$, consider the sets

$$N(i) = \{k \in I \mid k \succ_{j_0} i\}, \qquad J(i) = \{j \in J \mid i \succcurlyeq_j k \ \text{ for all } \ k \notin N(i)\}.$$

Observe that $J(i) \neq \varnothing$ since $j_0 \in J(i)$. If $N(i) = \varnothing$ then assume that $i \in I_{j_0}$. Suppose that $N(i) \neq \varnothing$. For every $k \in N(i)$, construct

$$J(k, i) = \{j \in J(i) \mid k \succ_j i\}.$$

Assume that $i \in I_{j_0}$ if

$$g_k > \sum_{j \in J(k,i)} p_{kj}$$

for every $k \in N(i)$, and that $i \notin I_{j_0}$ if there is $k \in N(i)$ for which the inequality is violated.

The meaning of the set I_j becomes clear in the following lemma which establishes that if the Leader plans to obtain profit by serving consumer $j \in J$ but opens no facilities in I_j then the Follower captures consumer j.

Given a fixed $I_0 \subset I$, for every $j \in J$, let $i_0(j)$ denote the element $i_0 \in I_0$ with $i_0 \succcurlyeq_j i$ for every $i \in I_0$. Given a $(0,1)$-vector $x = (x_i)$, $i \in I$, put $I_0(x) = \{i \in I \mid x_i = 1\}$. Given $(0,1)$-vectors $x = (x_i)$ and $y = (y_i)$, let $x \cup y$ denote the $(0,1)$-vector $z = (z_i)$ with $z_i = \max\{x_i, y_i\}$, $i \in I$.

Lemma 1. *For every admissible noncooperative solution $(X, \overline{Z})$, where $X = ((x_i), (x_{ij}))$ and $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$, and for $j_0 \in J$ with $p_{i_0 j_0} x_{i_0 j_0} > 0$ for some $i_0 \notin I_{j_0}$, we have*

$$\sum_{i \in I} \overline{z}_{ij_0} = 1.$$

Proof. Given $(0,1)$-vectors $x = (x_i)$ and $\overline{z} = (\overline{z}_i)$, consider the set $I_0 = I_0(x \cup \overline{z})$ and the elements $i_0(j)$, $j \in J$. Suppose that $p_{i_0 j_0} x_{i_0 j_0} > 0$ and $i_0 \notin I_{j_0}$ for some $j_0 \in J$, but the required equality fails. Consider the sets $N(i_0)$ and $J(i_0)$ and observe that $\overline{z}_i = 0$ for all $i \in N(i_0)$, while $i_0 = i_0(j)$ for every $j \in J(i_0)$. Since $i_0 \notin I_{j_0}$, there is $k \in N(i_0)$ for which there exists a set $J(k, i_0) \subset J(i_0)$ with

$$g_k \le \sum_{j \in J(k, i_0)} p_{kj}.$$

Given $k \in N(i_0)$, consider the sets

$$S_1(k) = \{j \notin J(i_0) \mid k \succ_j i_0(j),\ x_{i_0(j)} = 1\}, \qquad S_2(k) = \{j \notin J(i_0) \mid k \succ_j i_0(j),\ \overline{z}_{i_0(j)} = 1\}$$

and construct an admissible solution $\overline{Z}' = ((\overline{z}'_i), (\overline{z}'_{ij}))$ to F differing from the original admissible solution $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$ in that $\overline{z}'_k = 1$ and $\overline{z}'_{kj} = 1$ for $j \in J(k, i_0)$; $\overline{z}'_{kj} = 1$ for $j \in S_1(k)$; while $\overline{z}'_{kj} = 1$ and $\overline{z}'_{i_0(j)j} = 0$ for $j \in S_2(k)$.

Estimate the difference between the values of the objective function of F at $\overline{Z}'$ and $\overline{Z}$:

$$F(\overline{Z}') - F(\overline{Z}) = -g_k + \sum_{j \in J(k, i_0)} p_{kj} + \sum_{j \in S_1(k)} p_{kj} + \sum_{j \in S_2(k)} \left(p_{kj} - p_{i_0(j)j}\right).$$

Since p_{ij}, $i \in I$, are nonincreasing with respect to $\succ_j$, the last term is nonnegative; consequently, the difference is nonnegative. Hence, $\overline{Z}'$ is an optimal solution to F. Observe also that $j_0 \in J(k, i_0)$ implies

$$L(X, \overline{Z}') < L(X, \overline{Z})$$

in contradiction with $(X, \overline{Z})$ being an admissible noncooperative solution.

The proof of Lemma 1 is complete. □

Lemma 2. *For every admissible noncooperative solution $(X, \overline{Z})$, where $X = ((x_i), (x_{ij}))$ and $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$, and every $j \in J$, we have*

$$\left(\sum_{i \in I} p_{ij} x_{ij}\right)\left(1 - \sum_{i \in I} \overline{z}_{ij}\right) \le \max_{i \in I_j} p_{ij} x_i.$$

Proof. If $p_{ij} x_{ij} = 0$ for every $i \in I$ then the inequality holds. Suppose that $p_{i_0 j} x_{i_0 j} > 0$ for some $i_0 \in I$. If $i_0 \in I_j$ then the inequality holds as well, since

$$\sum_{i \in I} p_{ij} x_{ij} = p_{i_0 j} x_{i_0} \le \max_{i \in I_j} p_{ij} x_i.$$

However, if $i_0 \notin I_j$ then the inequality holds, since Lemma 1 yields $\sum_{i \in I} \overline{z}_{ij} = 1$.

The proof is complete. □

Define the matrix (h_{ij}), $i \in I$ and $j \in J$, by putting

$$h_{ij} = \begin{cases} 1, & \text{for } i \in I_j, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem. *The quantity*

$$\max_{(x_i)} \Big\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \max_{i | x_i = 1} p_{ij} h_{ij} \Big\}$$

is an upper bound for the optimal value of the objective function of Problem (L, F).

Proof. Every admissible noncooperative solution $(X, \overline{Z})$ with $X = ((x_i), (x_{ij}))$ and $\overline{Z} = ((\overline{z}_i), (\overline{z}_{ij}))$ satisfies

$$L(X, \overline{Z}) \leq -\sum_{i \in I} f_i x_i + \sum_{j \in J} \max_{i \in I_j} p_{ij} x_i = -\sum_{i \in I} f_i x_i + \sum_{j \in J} \max_{i | x_i = 1} p_{ij} h_{ij}$$

by Lemma 2 and the validity of

$$\max_{i \in I_j} p_{ij} x_i = \max_{i | x_i = 1} p_{ij} h_{ij}$$

for every $j \in J$. Hence,

$$\max_{(x_i)} \Big\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \max_{i | x_i = 1} p_{ij} h_{ij} \Big\}$$

is the required upper bound. The proof of the theorem is complete. □

This implies that the calculation of an upper bound reduces to solving the classical maximum facility location problem:

$$\max_{(x_i)(x_{ij})} \Big\{ -\sum_{i \in I} f_i x_i + \sum_{j \in J} \sum_{i \in I} p_{ij} h_{ij} x_{ij} \Big\},$$

$$\sum_{i \in I} x_{ij} \leq 1, \qquad j \in J,$$

$$x_i \geq x_{ij}, \qquad i \in I, \quad j \in J,$$

$$x_i, x_{ij} \in \{0, 1\}, \qquad i \in I, \quad j \in J.$$

If $X^* = ((x_i^*), (x_{ij}^*))$ is the optimal solution to this problem then, since p_{ij}, $i \in I$, are nonincreasing for fixed $j \in J$, we may assume that this is an admissible solution to L, which generates an admissible noncooperative solution $(X^*, \overline{Z})$ to Problem (L, F).

As a corollary of the theorem we point out some cases when the approximate solution $(X^*, \overline{Z})$ obtained while calculating an upper bound is an optimal noncooperative solution, while the upper bound is equal to the optimal value of the objective function of (L, F). Observe first of all that if X^* is the zero solution then the upper bound is equal to zero; and, consequently, $(X^*, \overline{Z})$ is the optimal noncooperative solution. Observe also that if $\overline{Z}$ is the zero solution then $(X^*, \overline{Z})$ is the optimal noncooperative solution since the value of the objective function of (L, F) at it is equal to the upper bound.

Thus, the *algorithm for calculating the upper bound* for the optimal value of the objective function of Problem (L, F) includes the two stages: At the first stage, we construct the matrix (h_{ij}), $i \in I$ and $j \in J$; while, at the second, we determine the optimal solution $((x_i^*), (x_{ij}^*))$ to the stated facility location problem.

The procedure for constructing the matrix (h_{ij}) consists of n steps. At step j, for fixed $j \in J$, we perform m similar substeps. At the beginning of substep i, put $h_{ij} = 1$. Then, for the set $I_0 = I \setminus (N(i)$, determine the elements $i_0(s)$, $s \in J$, and, for every $k \in N(i)$, calculate

$$-g_k + \sum_{s \mid i=i_0(s),\, k\succ_s i} p_{ks}.$$

If, for some $k \in N(i)$, this quantity is nonnegative then put $h_{ij} = 0$.

It is not difficult that the time complexity of one substep of the algorithm can be estimated as $O(mn)$. Thus, we can express the time complexity of the first stage of the algorithm as $O(m^2n^2)$.

At the second stage of the algorithm for solving the facility location problem, we can use a whole series of algorithms [1] based on the ideas of local search and implicit search, as well as commercially available software for solving integer linear programming problems.

3. ALGORITHMS FOR CONSTRUCTING APPROXIMATE SOLUTIONS TO PROBLEM (L, F)

We noted above that as an approximate solution to (L, F) we may consider every admissible solution X to L. The value of the objective function of (L, F) at the corresponding admissible noncooperative solution $(X, \overline{Z})$ is uniquely determined by X. Meanwhile, the optimal noncooperative solution $\overline{Z}$ to F is constructed as a result of solving F and an auxiliary problem (10)–(14). In addition, the admissible solution $X = ((x_i), (x_{ij}))$ to L itself is uniquely determined by the $(0, 1)$-vector (x_i). Every vector of this type uniquely determines the value $L(X, \overline{Z})$ of the objective function of (L, F) at the corresponding admissible noncooperative solution $(X, \overline{Z})$. For this reason, we may regard (L, F) as the problem of maximizing some pseudo-Boolean function $f(x)$, $x \in B^m$. The value of this functions at an arbitrary $(0, 1)$-vector x is equal to the value of the objective function of Problem (L, F) at the admissible noncooperative solution $(X, \overline{Z})$ constructed from the given $(0, 1)$-vector x.

Since, simultaneously with the calculation of an upper bound for the optimal value of the objective function of (L, F), we determine an admissible solution $X^* = ((x_i^*), (x_{ij}^*))$ of Problem L, we may regard the $(0, 1)$-vector (x_i^*) as an initial approximate solution to the problem of maximizing a pseudo-Boolean function $f(x)$.

Consider algorithms for improving an initial approximate solution constructed basing on the standard local search procedure [1, 9]. These algorithms produce a locally optimal solution in a given neighborhood. In the case of the problem of optimizing a pseudo-Boolean function $f(x)$, $x \in B^m$, the following neighborhoods are often used:

$$N_1(x) = \{y \in B^m \mid d(x, y) = 1\},$$

$$N_2(x) = \{y \in B^m \mid d(x, y) = 2,\ d(0, x) = d(0, y)\},$$

$$N_3(x) = N_1(x) \cup N_2(x),$$

where $d(x, y)$ is the Hamming distance equal to the number of differing components of the $(0, 1)$-vectors x and y. The elements of the neighborhood $N_1(x)$ result from x by changing the value of one component, while those of $N_2(x)$, by changing two components whose sum is equal to 1.

In our case, local search in the "wide" neighborhood $N_3(x)$ can turn out too laborious even since, for every element of the neighborhood, the evaluation of $f(x)$ requires solving two integer linear programming problems. Thus, given some $(0, 1)$-vector x, define a neighborhood $N_0(x) \subset N_3(x)$ containing a relatively small number of *essential* variants of adjusting the current solution x and construct local search algorithms with $N_0(x)$.

Take the current solution $x = (x_i)$. Consider the *profitability* of facility $k \in I$ opened by the Leader as the main indicator using which we construct essential adjustments to the current solution x. Suppose that $k \in I_0(x)$. For the set $I_0 = I_0(x)$, define the elements $i_0(j)$, $j \in J$, and consider

$$\Delta_k(x) = -f_k + \sum_{j \mid i_0(j)=k} p_{kj}$$

which is called the *profitability* of facility k with respect to x.

Using the concept of profitability, select the set $N_0(x) \subset N_3(x)$ of essential variations of the current solution $x = (x_i)$. Construct the neighborhood $N_0(x)$ consisting of the $(0,1)$-vectors $x^k = \big(x_i^k\big)$, $k \in I$, as follows:

If $x_k = 1$ then put $x_i^k = x_i$ for $i \neq k$ and $x_k^k = 0$ and declare x^k constructed.

If $x_k = 0$ then put $x_i^k = x_i$ for $i \neq k$ and $x_k^k = 1$.

Then we calculate $\Delta_k(x^k)$. Two cases are possible: $\Delta_k(x^k) \geq 0$ and $\Delta_k(x^k) < 0$. In the first case, when the profitability of facility k is nonnegative, calculate $\Delta_i(x^k)$ for every $i \in I_0(x)$. If $\Delta_i(x^k) \geq 0$ for all $i \in I_0(x)$, i.e., opening the new facility k does not lead to some old facilities opened by the Leader becoming unprofitable, then declare x^k constructed. However, if $\Delta_i(x^k) < 0$ for some $i \in I_0(x)$ then determine $i_0 \in I_0(x)$ with $\Delta_{i_0}(x^k) \leq \Delta_i(x^k)$ for every $i \in I_0(x)$, put $x_{i_0}^k = 0$, and declare x^k constructed.

If $\Delta_k(x^k) < 0$, i.e., the profitability of the new facility is negative, then among the old facilities find the one whose removing increases the profitability of the new facility k the most. To this end, for every $l \in I_0(x)$, construct the $(0,1)$-vector $x^{kl} = \big(x_i^{kl}\big)$, where $x_i^{kl} = x_i^k$ for $i \neq l$ and $x_l^{kl} = 0$, and calculate $\Delta_k(x^{kl})$. Then determine $l_0 \in I_0(x)$ with $\Delta_k(x^{kl_0}) \geq \Delta_k(x^{kl})$ for all $l \in I_0(x)$. If $\Delta_k(x^{kl_0}) \geq 0$ then put $x_{l_0}^k = 0$ and declare x^k constructed.

The *algorithm for improving the original approximate solution* $x^* = (x_i^*)$ that is obtained in result of calculating the upper bound amounts to a local search procedure with the neighborhood $N_0(x)$ starting with the vector x^* until we find a local maximum of $f(x)$. The algorithm consists of a preliminary step and a number of similar main steps.

At the preliminary step, there is an initial approximate solution $x = x^*$. The step amounts to calculating the value L of $f(x)$ at the solution x.

At each main step, there is a solution x and the value L of $f(x)$ at x. The step amounts to constructing successively for every $k \in I$ some vectors x^k and calculating the values L^k of $f(x)$ at x^k. Simultaneously we determine a solution x^{k_0}, $L^{k_0} > L$, *improving* the current solution x. If we fail to find it then the algorithm halts, and x is the required approximate solution. Otherwise, put $L = L^{k_0}$ and $x = x^{k_0}$; and then start the next step.

In the presented general scheme of the algorithm, the rule for choosing the improving solution x^{k_0} in the neighborhood $N_0(x)$ needs more details. We consider four methods for choosing $k_0 \in I$. In the first method, we inspect all $k \in I$ and choose $k_0 \in I$ with $L^{k_0} > L$ and $L^{k_0} \geq L^k$ for every $k \in I$. The other three methods amount to inspecting the elements of I in a certain order and choosing as k_0 the first $k \in I$ with $L^k > L$. In the second method, the elements of I are inspected in the order of increasing k; in the third method, $k \in I$ are taken in the decreasing order of the number of occurrences of $k \in I$ in the sets I_j, $j \in J$, determined while calculating the upper bound. In the fourth method, the order of inspection of I is random, but some $k \in I$ are forbidden. The list of forbidden elements is formed from $k \in I$ chosen at the previous steps, and includes at most $m/3$ elements.

In closing, we present the results of simulation with the proposed algorithms for constructing approximate solutions to Problem (L,F) differing in the method for choosing the best elements in a neighborhood. We index the algorithms with 1, 2, 3, and 4 in accordance with the methods used for choosing the improving solutions. The computations were carried out for Problem (L,F) on a network [2], i.e., for the problem, where, firstly, the set I of possible facility locations and the set J of consumers coincide with the set of vertices of some graph; and the order relation for each consumer is determined by the lengths of the shortest paths from the corresponding vertices to the other vertices.

The original data in our examples of Problem (L,F) on a network is formed randomly as follows: in the unit square, we take n random points which constitute the vertices of the graph. The distance r_{ij} between vertices i and j is calculated as the distance between these points on the plane. The presence of an edge between vertices i and j is determined randomly by giving a probability of an edge appearing. These quantities are specified so that the number of edges in the graph lie in the interval from $1.5n$ to $2n$.

We calculate p_{ij}, $i \in I$, $j \in J$, using

$$p_{ij} = \begin{cases} b_j, & \text{if } \ d_{ij} \leq d_j, \\ 0, & \text{otherwise,} \end{cases}$$

where d_{ij} is the length of the shortest path from vertex i to vertex j; d_j is a parameter; b_j is a uniformly distributed random variable taking the integer values in the interval from b_0 to b^0. In the examples,

$$d_j = 0.7, \qquad b_0 = 10, \qquad b^0 = 20.$$

The quantities f_i, $i \in I$, are equal to 40, while each of g_i, $i \in I$, is a uniformly distributed random variable taking the integer values from 25 to 35.

We carried out computations for a series of 40 problems for the fixed values $n = 20, 30, 40,$ and 50. For the four algorithms considered, the table below includes the averages over the series of problems of the following quantities:

UB, the upper bound;

VL, the value of the objective function of the Leader's problem on the approximate solution;

$|x|$, the number of facilities opened by the Leader and determined by the approximate solution;

$|z|$, the number of facilities opened by the Follower and determined by the approximate solution;

N, the number of steps of the algorithm to reach the approximate solutions;

IP, the number of linear integer programming problems solved;

UB/VL, the estimated relative precision of the resulting approximate solutions, equal to the ratio of UB and VL.

In addition, for $n = 20$ and $n = 30$, the table includes the average values of the following quantities:

VL^*, the optimal value of the objective function of the Leader's problem obtained by the implicit search procedure;

UB/VL^*, the relative precision of the computed upper bound, equal to the ratio of UB and VL^*;

VL^*/VL, the relative precision of the obtained approximate solution, equal to the ratio of VL^* and VL.

From the table it is clear that the average relative precision of the approximate solutions is roughly the same for all algorithms under consideration. The first algorithm is the most preferable in this sense, while the second one is the least. However, the advantage of the first algorithm is not absolute since, for some examples, the third algorithm yields the best solution.

The average value of the upper bound, as the table implies, exceeds the optimal value of the objective function almost by a factor of two. This precision is not satisfactory to obtain a priori estimates for the precision of the approximate solutions. At the same time, the initial approximate solution obtained simultaneously with the upper bound is a good starting point for constructing a local maximum of the objective function. The table implies that on average 3–7 steps suffice for that. The resulting first locally optimal solution in many examples is either optimal or close to optimal in the value of the objective function. Thus, we may conclude that the precision of the approximate solution obtained by the considered algorithms is quite satisfactory.

Note that the search for good approximate solutions certainly must not finish at finding the first locally optimal solution; and consequently, these algorithms possess a real potential for successful modification.

In the course of simulations we also compared the neighborhood $N_0(x)$ and the wider neighborhood $N_3(x)$. We observed that, in more than 85% of cases, the locally optimal solution in $N_0(x)$ is also locally optimal in $N_3(x)$. This means that, despite the set $N_3(x)$ has considerably more elements than $N_0(x)$, in the majority of cases, the element of $N_3(x)$ on which the value of the objective function is the greatest lies in $N_0(x)$ as well.

n	Algorithm	UB	VL	$\lvert x\rvert$	$\lvert z\rvert$	N	IP	$\frac{UB}{VL}$	VL^*	$\frac{UB}{VL^*}$	$\frac{VL^*}{VL}$
20	1	84.73	37.50	3.18	1.85	2.80	116.00	2.26.	42.55	1.99	1.13
	2	84.73.	36.20	3.03.	1.78	3.80	85.05	2.34	42.55	1.99	1.18
	3	84.73	37.25	3.13	1.85	3.40	68.40	2.27	42.55	1.99	1.14
	4	84.73	36.78	3.20	1.85	3.53	132.00	2.30	42.55	1.99	1.16
30	1	132.45	58.68	4.80	2.60	3.57	218.50	2.26	67.90	1.95	1.16
	2	132.45	54.32	4.75	2.68	4.35	135.05	2.44	67.90	1.95	1.25
	3	132.45	55.87	4.75	2.60	4.35	109.85	2.37	67.90	1.95	1.22
	4	132.45	54.72	4.90	2.60	4.47	196.00	2.42	67.90	1.95	1.24
40	1	198.05	87.88	7.05	3.40	4.70	380.00	2.25			
	2	198.05	84.40	6.80	3.40	6.15	225.90	2.35			
	3	198.05	85.70	6.65	3.58	6.10	177.95	2.31			
	4	198.05	82.28	7.05	3.65	6.00	260.00	2.41			
50	1	240.20	104.13	7.90	4.08	4.82	486.50	2.31			
	2.	240.20	101.53	7.80	3.98	7.15	315.65	2.37			
	3	240.20	101.03	7.43	4.04	7.05	249.10	2.38			
	4	240.20	100.43	7.78	4.08	6.95	324.00	2.39			

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