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Upper Bounds for Objective Functions of Discrete Competitive Facility Location Problems

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Abstract—Under study is the problem of locating facilities when two competing companies successively open their facilities. Each client chooses an open facility according to his own preferences and return interests to the leader firm or to the follower firm. The problem is to locate the leader firm so as to realize the maximum profit (gain) subject to the responses of the follower company and the available preferences of clients. We give some formulations of the problems under consideration in the form of two-level integer linear programming problems and, equivalently, as pseudo-Boolean two-level programming problems. We suggest a method of constructing some upper bounds for the objective functions of the competitive facility location problems. Our algorithm consists in constructing an auxiliary pseudo-Boolean function, which we call an *estimation function*, and finding the minimum value of this function. For the special case of the competitive facility location problems on paths, we give polynomial-time algorithms for finding optimal solutions. Some results of computational experiments allow us to estimate the accuracy of calculating the upper bounds for the competitive location problems on paths.

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INTRODUCTION

The uncapacitated facility location problem is a well-known discrete optimization problem [3, 11]. In the maximization facility location problem, a manufacturer decides which facilities from a given set to open and which open facility to assign to each client so as to maximize profit. The profit is equal to the total benefit that the open facilities receive from serving the assigned clients minus some fixed costs of opening facilities. Although we have two parts (i.e., the manufacturer and the consumer), it is supposed in this model that a decision is made by the manufacturer that opens facilities and assigns clients to the open facilities according to his objective function.

In this paper, we study a more general model, namely, the locating (opening) facilities under the assumption of competition. It is supposed that two rival firms, manufacturing some product, successively make decisions of opening their facilities from the given sets of possible facilities. In this model, each client is assumed to make decision on the basis of his own preferences: he selects a best open facility giving the profit to one of the firms. By analogy with the Stackelberg model [15], the process of making decision in this competition location model consists of the three stages: At the first stage, one of the firms (the *leader firm*) taking into account the possible responses of the second firm (the *follower firm*) opens (locates) its facilities. At the second stage, the follower firm having information about open facilities of the leader firm opens its facilities. Finally, each client selects a best open facility on the basis of his own preferences. The literature, where the decision making process or its part is formalized as a three-stage optimization process with using integer and mixed programming problems, can be considered as vast. We will not try to overview all publications but refer to the definitive monographs and surveys on the subject [6, 9, 11, 12, 14]. Note that almost all competitive location models use the equality constraint on the number of the facilities opened both by the leader and follower firms. In this connection, note

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that the authors of [8] suggests a three-stage optimization model that has no bounds on the number of open facilities but contains some fixed costs for opening facilities by both the leader and follower firms and which, therefore, can be considered as a generalization of the classical uncapacitated facility location problem. Note also that in all three-stage optimization models the objective function is the profit (gain). Still different notions of optimal solutions are considered. For instance, [8] introduces the notion of *competitively stable solution*, i.e., a solution in which the follower firm can open no "viable" facility. Completing the survey on the available literature on the competition facility location problems, we may conclude that, despite a vast number of papers on the subject containing different settings of the problems and some studies of the properties of solutions to these problems, the number of papers is not great where some methods of constructing the optimal solutions and upper bounds on the objective functions are suggested. Note the papers [4, 13] that study the competitive facility location problems under the auxiliary assumption that the number of the facilities opened by the follower firm is not great or even equals one.

In this paper, we present and study the problems that formalize the above three-stage process of competitive location of the leader and the follower firm. The models contain the fixed costs for opening facilities, and expenses of opening the same facility can be different for the leader and the follower firm. The preferences of the clients are formalized as in the location problems with orders [1, 5], by assigning to each client a linear order on the set of possible facilities. This results in two-level mathematical programming problems [7]. In the problems of this type, the constraints for part of variables are not in the form of explicit relations (equations or inequalities) but are given implicitly as the set of optimal solutions to some "inner" optimization problem whose parameters depend on a distinct part of variables.

In the next section, we give the formulations of the competitive location problems in the form of two-level mathematical programming problems and describe the notion of optimal solution for these problems. We consider the two settings that differ by the objective functions of the inner problem (the follower firm problem): in the first, the profit is maximized; while in the next, the income of the follower firm. We consider the different special cases of the problems in which the profit realized by a facility does not depend on the facility but just on the clients that selected this facility. These models are written as the two-level integer linear programming problems and are the main object of further study. The possibility is mentioned of a representation of the second problem as a max-min integer linear programming problem. In Section 2, we give some equivalent formulations of the problems in the form of pseudo-Boolean two-level programming problems [10]. Section 3 considers the competitive location problems on networks. For these problems, the order relations are determined by the shortest distances between the vertices of a network. It is shown that, in the case when the network is a path, the problems under consideration are polynomial-time solvable. The corresponding algorithms constructed on the basis of the dynamic programming method [2, 3] run in time $O(n^5)$, where n is the number of vertices of the network under study. In Section 4, we present an algorithm for finding an upper bound on the values of the objective functions of our competitive location problems. Our approach uses their statements in the form of pseudo-Boolean programming problems and reduces to constructing a pseudo-Boolean function called an *estimation function* and calculating its minimum value. In the concluding section, we consider a numerical instance of the competitive location problem for which the estimate function is constructed and its upper bound is calculated. We present some results of a computational experiment with the algorithm of constructing the upper bound for some classes of competitive location problems on a network that alow us to evaluate the accuracy of calculating the upper bound for the problems of these classes.

1. FORMULATION OF COMPETITIVE LOCATION PROBLEMS

We now formulate the problems that generalize the classical maximization facility location problem and, as was mentioned above, describe the situation when two competitive firms (the leader firm and the follower firm) successively open their facilities manufacturing some product for satisfying the demands of a given set of consumers (clients). The set of possible sites is given for locating facilities at each of which either of the firms can open its own facility. For each of these sites, some fixed costs of opening a facility are known, which can be different for the leader firm and the follower firm. It is assumed that each client selects a serving facility on the basis of his own preferences, which can rank (order) all facilities available to be opened. The rule of selecting a serving facility by the client consists in selecting the first open facility according to the given order. We assume that, for each facility and each client, we are given the value of the profit that this facility realizes when serving this client. The income of either firm is the sum of the profits received by its open facilities; and the profit of a firm is equal to the income minus the fixed costs for opening the facilities.

In the situation under consideration, the decisions that determine the incomes and the profits of the firms are made by all participants including the leader firm, the follower firm, and the clients. The decision making process can be represented as the following three stage process:

1. The leader firm opens its facilities of possible sites of their location subject to the information that the follower firm can also open its facilities and "capture" part of clients.

2. The follower firm, informed about the open facilities of the leader firm, opens its own facilities at the sites not occupied by the open facilities of the leader firm.

3. Each client having information about the set of open facilities of either firm selects a serving facility according to his rules and returns interest either to the leader firm or to the follower firm.

The problem stated on behalf of the leader company consists in selecting the location of facilities so as to realize the maximum profit provided that the follower firm "captures" a part of clients by opening its facilities according to its objective function. It is assumed that the goal of the follower firm is known and both firms are informed about the rules that the clients use to select serving facilities.

Consider the two settings of the problem that differ in the objective functions of the follower firm: In the first, we assume that the goal of the leader firm as well as the follower firm is the realization of the maximum profit. In the second, the goal of the leader firm is to realize the maximum income, i.e., capturing the maximum number of clients. Moreover, in the second case, we will also assume that each facility opened by the follower firm cannot be detrimental, i.e., the income realized by this facility cannot be smaller than the fixed cost for its opening.

To formalize the statements of the problems we introduce the following notation:

$I = \{1, \dots, m\}$ is the set of facilities (possible sites for locating them);

$J = \{1, \dots, n\}$ is the set of clients;

p_{ij} is the income realized by facility i in I opened by the leader firm when serving client j in J;

q_{ij} is the income realized by facility i opened by the follower firm when serving client j in J;

$\prec_j$ is a linear order on I determining the preferences of client j in J, and $i \prec_j k$ means that of the two open facilities i and k in I client j selects facility i; the relation $i \preccurlyeq_j k$ means that either $i \prec_j k$ or $i = k$; in the cases when it is clear with respect to which client the facilities are compared the index j in $\prec_j$ will be omitted;

f_i is the fixed cost of the leader firm for opening facility i in I;

g_i is the fixed cost of the follower firm for opening facility i in I.

To formally write the problems we use the variables of the classical incapacitated facility location problem:

x_i is the variable indicating if facility i in I is opened by the leader firm, i.e., $x_i = 1$ if it is opened and $x_i = 0$ otherwise.

x_{ij} is the variable indicating if facility i in I opened by the leader firm is selected by client j in J, i.e., $x_{ij} = 1$ if it is selected and $x_{ij} = 0$ otherwise;

z_i is the variable indicating if the follower firm opens facility i in I, i.e., $z_i = 1$ if it opens and $z_i = 0$ otherwise;

z_{ij} is the variable indicating if client j in J selects facility i in I opened by the follower firm, i.e., $z_{ij} = 1$ if it selects and $z_{ij} = 0$ otherwise.

By using the above variables, in the case when the goal of the follower firm is realizing the maximum profit, the competitive location problem is written as follows:

$$\max_{(x_i),(x_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} \Big(\sum_{i\in I} p_{ij} x_{ij}\Big)\Big(1 - \sum_{i\in I} \widetilde{z}_{ij}\Big)\Big\}, \tag{1}$$

$$\sum_{i\in I} x_{ij} = 1, \qquad j \in J, \tag{2}$$

$$x_i \ge x_{ij}, \qquad i \in I,\ \ j \in J, \tag{3}$$

$$x_i + \sum_{i \prec_j l} x_{lj} \le 1, \qquad i \in I,\ \ j \in J, \tag{4}$$

$$x_i, x_{ij} \in \{0,1\}, \qquad i \in I,\ \ j \in J, \tag{5}$$

$$(\widetilde{z}_i), ((\widetilde{z}_{ij})) \text{ is an optimal solution of the problem (7)–(11),} \tag{6}$$

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} q_{ij} z_{ij} \Big\}, \tag{7}$$

$$\sum_{i\in I} z_{ij} \le 1, \qquad j \in J, \tag{8}$$

$$z_i \ge z_{ij}, \qquad i \in I,\ \ j \in J, \tag{9}$$

$$x_i + z_i + \sum_{i \prec_j l} z_{lj} \le 1, \qquad i \in I,\ \ j \in J, \tag{10}$$

$$z_i, z_{ij} \in \{0,1\}, \qquad i \in I,\ \ j \in J. \tag{11}$$

The above problem like any two-level programming problem includes the *inner* optimization problem (7)–(11) which we call the *follower firm problem*.

The objective function (1) of the problem expresses the value of the profit realized by the leader firm subject to the loss of part of the clients “captured” by the follower firm. It is indeed the case as the inequality (8) holds for some $j \in J$ with equality; i.e., if, for a client j, there is a facility opened by the follower firm that is better than any open facility of the leader firm then the income realized by the leader firm from client j becomes equal to zero. The constraint (2) guarantees that each client can select only one facility of the leader firm; and the inequality (3) means that only one open facility can be selected. The constraints (4) realize the rule used by a client to select an open facility. The objective function and the constraints of the problem (7)–(11) can be treated similarly. The constraint (10) guarantees selecting an open facility by a client according to the given rule and, moreover, shows that if a facility is opened by the leader firm then it cannot be opened by the follower firm.

In the case when the goal of the follower firm is realizing the maximum income subject to the additional assumption of profitability of each open facility, the competitive location problem differ only in the constraints related to the follower firm problem. These are written as follows:

$$((\widetilde{z}_i), (\widetilde{z}_{ij})) \text{ is an optimal solution of the problem (13)–(18),} \tag{12}$$

$$\max_{(z_i),(z_{ij})} \sum_{j\in J}\sum_{i\in I} q_{ij} z_{ij}, \tag{13}$$

$$\sum_{i\in I} z_{ij} \le 1, \qquad j \in J, \tag{14}$$

$$z_i \ge z_{ij}, \qquad i \in I,\ \ j \in J, \tag{15}$$

$$x_i + z_i + \sum_{i \prec_j l} z_{lj} \le 1, \qquad i \in I,\ \ j \in J, \tag{16}$$

$$\sum_{j\in J} q_{ij} z_{ij} \geq g_i z_i, \qquad i \in I, \tag{17}$$

$$z_i, z_{ij} \in \{0,1\}, \qquad i \in I,\ j \in J. \tag{18}$$

Here (17) means that the income realized by each facility opened by the follower firm must be at least the fixed cost for opening this facility.

Let us specify the notion of optimal solution of the competitive facility location problems (1)–(11) and (1)–(5), (12)–(18) taking it into account that the inner problems can have several different optimal solutions.

Let X denote the solution $((x_i), (x_{ij}))$ satisfying (2)–(5) which we call the *feasible solution* of the competitive facility location problem (1)–(11) and (1)–(5), (12)–(18). For a fixed solution X, let Z denote a feasible solution $((z_i),\ (z_{ij}))$ of the problem (7)–(11) or (13)–(18) depending on which problem is the inner problem of the competitive facility location problem under consideration. Let $O(X)$ denote the set of optimal solutions $\widetilde{Z}$ of the inner problem. Let $L(X, Z)$ denote the value of the objective function (1) on X and Z. Then the above-formulated competitive facility location problems can be briefly written as

$$\max_X L(X, \widetilde{Z}), \qquad \widetilde{Z} \in O(X).$$

For the competitive facility location problem, call a feasible solution X^* *optimal* if the inner problem has an optimal solution $\widetilde{Z}^* \in O(X)$ satisfying the following two conditions:

1. $L(X^*, \widetilde{Z}^*) \leq L(X^*, \widetilde{Z})$ for every $\widetilde{Z} \in O(X^*)$.
2. For every feasible solution X, there exists an optimal solution $\widetilde{Z} \in O(X)$ such that

$$L(X^*, \widetilde{Z}^*) \geq L(X, \widetilde{Z}).$$

It is easy that if, for every solution X, the values of $L(X, \widetilde{Z})$ are the same for each $\widetilde{Z} \in O(X)$ then the first optimality condition certainly holds. In the general case, this definition of optimal solution gives that the competitive facility location problems (1)–(11) and (1)–(5), (12)–(18) are equivalently rewritten in the form of the problem

$$\max_X \min_{\widetilde{Z}\in O(X)} L(X, \widetilde{Z}).$$

This is the max-min two-level mathematical programming problem such that the set of feasible solutions $\widetilde{Z}$ is determined implicitly as the set of optimal solutions of the inner problem.

We will study the competitive facility location problems (1)–(11) and (1)–(5), (12)–(18) under the following additional condition that allows us to restate them as the two-level integer linear programming problems: Assume that the income realized by each facility from client j in J does not depend on the facility and is equal to b_j; i.e., we assume that, for every $j \in J$ and $i \in I$, we have $p_{ij} = q_{ij} = b_j$.

In this case, the function $L(X, Z)$ takes the form

$$L(X, Z) = -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \Big(1 - \sum_{i\in I} \widetilde{z}_{ij}\Big)$$

and, therefore, the variables x_{ij}, $i \in I$ and $j \in J$, can be excluded from the problems. In result, the problem (1)–(11) can be rewritten as

$$\max_{(x_i)} \min_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \big(1 - \sum_{i\in I} \widetilde{z}_{ij}\big)\Big\}, \tag{19}$$

$$\sum_{i\in I} x_i \geq 1, \tag{20}$$

$$x_i \in \{0,1\}, \qquad i \in I, \tag{21}$$

$$((\widetilde{z}_i), (\widetilde{z}_{ij})) \text{ is an optimal solution of the problem (23)–(27),} \tag{22}$$

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} b_j z_{ij} \Big\}, \tag{23}$$

$$\sum_{i\in I} z_{ij} \le 1, \qquad j \in J, \tag{24}$$

$$z_i \ge z_{ij}, \qquad i \in I,\ \ j \in J, \tag{25}$$

$$x_i + z_i + \sum_{i \prec_j l} z_{lj} \le 1, \qquad i \in I,\ \ j \in J, \tag{26}$$

$$z_i, z_{ij} \in \{0, 1\}, \qquad i \in I,\ \ j \in J. \tag{27}$$

The similar formulation of the problem (1)–(5), (12)–(18) differs from (19)–(27) in the constraints related to the follower firm problem which have the following form:

$$((\widetilde{z}_i), (\widetilde{z}_{ij})) \text{ is an optimal solution of the problem (29)–(34),} \tag{28}$$

$$\max_{(z_i),(z_{ij})} \sum_{j\in J}\sum_{i\in I} b_j z_{ij}, \tag{29}$$

$$\sum_{i\in I} z_{ij} \le 1, \qquad j \in J, \tag{30}$$

$$z_i \ge z_{ij}, \qquad i \in I,\ \ j \in J, \tag{31}$$

$$x_i + z_i + \sum_{i \prec_j l} z_{lj} \le 1, \qquad i \in I,\ \ j \in J, \tag{32}$$

$$\sum_{j\in J} b_j z_{ij} \ge g_i z_i, \qquad i \in I, \tag{33}$$

$$z_i, z_{ij} \in \{0, 1\}, \qquad i \in I,\ \ j \in J. \tag{34}$$

The above formulations (19)–(27) and (19)–(21), (28)–(34) of the competitive facility location problems are the two-level integer linear programming problems. Nevertheless, this remark does not apply to either problem in the same manner.

Given a fixed solution X of the problems (19)–(27) and (19)–(21), (28)–(34), let $O_1(X)$ denote the set of optimal solutions of the problem (23)–(27); and let $O_2(X)$ and $D_2(X)$ be the set of optimal and feasible solutions of the problem (29)–(34), respectively. Let $F_2(Z)$ be the objective function (29) of the problem (29)–(34). Note that since

$$L(X, Z) = -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \Big(1 - \sum_{i\in I} z_{ij}\Big) = -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j - F_2(Z),$$

the value of $L(X, Z)$ will be the same for all optimal solutions $\widetilde{Z} \in O_2(X)$. Therefore,

$$\begin{aligned}\min_{\widetilde{Z}\in O_2(X)} L(X, \widetilde{Z}) &= -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j - F_2(\widetilde{Z}) \\ &= -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j - \max_{Z\in D_2(X)} F_2(Z) = \min_{Z\in D_2(X)} L(X, Z).\end{aligned}$$

Hence, the problem (19)–(21), (28)–(34) is equivalent to the following max-min integer linear programming problem:

$$\max_{(x_i)} \min_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \Big(1 - \sum_{i\in I} z_{ij}\Big)\Big\},$$

$$\sum_{i\in I} x_i \geq 1, \qquad x_i \in \{0,1\}, \ \ i \in I,$$

where $((z_i),(z_{ij}))$ is a feasible solution of (29)–(34).

In this problem, the set of solutions $((z_i),(z_{ij}))$ of which the best one is selected is given explicitly by the constraints (30)–(34).

Taking into account this representation of the problem (19)–(21), (28)–(34), we see that

$$\min_{\widetilde{Z}\in O_2(X)} L(X,\widetilde{Z}) = \min_{Z\in D_2(X)} L(X,Z) \leq \min_{\widetilde{Z}\in O_1(X)} L(X,\widetilde{Z}).$$

for every solution X of the problems (19)–(27) and (19)–(21), (28)–(34). The last inequality holds because, for an optimal solution $\widetilde{Z}$ of (23)–(27), the constraint (33) is clearly satisfied. Therefore, the solution $\widetilde{Z}$ is a feasible solution of (29)–(34).

It follows that an upper bound on the values of the objective function (19) of the problem (19)–(27) will be simultaneously an upper bound for the values of the objective function (19) of the problem (19)–(21), (28)–(34).

2. EQUIVALENT FORMULATION OF THE COMPETITIVE FACILITY LOCATION PROBLEMS

Since, for a fixed solution (x_i) of the problems (19)–(27) and (19)–(21), (28)–(34), the values of the variables z_{ij}, $i \in I$ and $j \in J$, of a feasible solution $((z_i),(z_{ij}))$ of the inner problems (23)–(27) and (29)–(34) are determined uniquely by the vectors (x_i) and (z_i); therefore, these variables can be excluded from consideration as well as previously were excluded the variables x_{ij}, $i \in I$ and $j \in J$. To obtain the corresponding equivalent formulations of our problems introduce the following notation:

For an arbitrary $(0,1)$-vector $w = (w_i)$, $i \in I$, and for given $j \in J$, let $i_j(w)$ denote i_0 in the set $I_0 = \{i \in I \mid w_i = 0\}$ such that $i_0 \preccurlyeq_j i$ for every $i \in I_0$. If $I_0 = \varnothing$ then let $i_j(w)$ denote $i_0 \in I$ such that $i \preccurlyeq_j i_0$ for every $i \in I$. For $(0,1)$-vectors (x_i) and (z_i), let $y = (y_i)$ and $u = (u_i)$ denote $(0,1)$-vectors such that $y_i = 1 - x_i$ and $u_i = 1 - z_i$, $i \in I$.

Using the above notation, we obtain that, for every solutions (x_i) and $((z_i),(z_{ij}))$ for every $j \in J$, the following equalities hold:

$$\sum_{i\in I} z_{ij} = 1 - \prod_{i \prec i_j(y)} u_i = \prod_{i \preccurlyeq i_j(u)} y_i.$$

Indeed, assume that $u_i = 1$ for every $i \prec i_j(y)$. Then $i_j(y) \preccurlyeq i_j(u)$ and so $y_i = 0$ for some $i \preccurlyeq i_j(u)$. Consequently, in this case, the equalities hold. If $u_i = 0$ for some $i \prec i_j(y)$ then $i_j(u) \prec i_j(y)$ and so $y_i = 1$ for every $i \preccurlyeq i_j(u)$. Therefore, in this case, the desired equalities hold as well.

Using these equalities, we have

$$\max_{(x_i)} \min_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \Big(1 - \sum_{i\in I} \widetilde{z}_{ij}\Big)\Big\}$$
$$= \max_{(y_i)} \min_{(\widetilde{u}_i)} \Big\{ -\sum_{i\in I} f_i (1 - y_i) + \sum_{j\in J} b_j \Big(1 - \prod_{i \preccurlyeq i_j(\widetilde{u})} y_i\Big)\Big\}$$

$$= -\sum_{i\in I} f_i + \sum_{j\in J} b_j + \max_{(y_i)} \min_{(\widetilde{u}_i)} \Big\{ \sum_{i\in I} f_i y_i - \sum_{j\in J} b_j \prod_{i \preccurlyeq i_j(\widetilde{u})} y_i \Big\}$$

$$= -\sum_{i\in I} f_i + \sum_{j\in J} b_j - \min_{(y_i)} \max_{(\widetilde{u}_i)} \Big\{ -\sum_{i\in I} f_i y_i + \sum_{j\in J} b_j \prod_{i \preccurlyeq i_j(\widetilde{u})} y_i \Big\},$$

$$\max_{(z_i),(z_{ij})} \Big\{ -\sum_{i\in I} g_i z_i + \sum_{j\in J}\sum_{i\in I} b_i z_{ij} \Big\}$$

$$= \max_{(u_i)} \Big\{ -\sum_{i\in I} g_i (1-u_i) + \sum_{j\in J} b_j \Big(1 - \prod_{i \prec i_j(y)} u_i\Big) \Big\}$$

$$= -\sum_{i\in I} g_i + \sum_{j\in J} b_j - \min_{(u_i)} \Big\{ -\sum_{i\in I} g_i u_i + \sum_{j\in J} b_j \prod_{i \prec i_j(y)} u_i \Big\}.$$

Whence we obtain the equivalent formulation of the problem (19)–(27) in the form of the min-max pseudo-Boolean two-level programming problem:

$$\min_{(y_i)} \max_{(\widetilde{u}_i)} \Big\{ -\sum_{i\in I} f_i y_i + \sum_{j\in J} b_j \prod_{i \preccurlyeq i_j(\widetilde{u})} y_i \Big\}, \tag{35}$$

$$\prod_{i\in I} y_i = 0, \tag{36}$$

$$y_i \in \{0,1\}, \qquad i \in I, \tag{37}$$

$$(\widetilde{u}_i) \text{ is an optimal solution of the problem (39)–(40)}, \tag{38}$$

$$\min_{(u_i)} \Big\{ -\sum_{i\in I} g_i u_i + \sum_{j\in J} b_j \prod_{i \prec i_j(y)} u_i \Big\}, \tag{39}$$

$$u_i \in \{0,1\}, \qquad i \in I. \tag{40}$$

By analogy, we obtain that the problem (19)–(21), (28)–(34) has an equivalent formulation as a min-max pseudo-Boolean programming problem.

3. NETWORK COMPETITIVE FACILITY LOCATION PROBLEMS

Let $G = (V, E)$ be a connected graph with vertex set V and edge set E, and let each edge have positive weight called the *edge length*. The length of a path in G is equal to the sum of lengths of the edges constituting this path. Call a path from vertex i to vertex j *shortest* if the length of this path is at most the length of any path from i to j. Denote the length of such a path by $d(i, j)$. We assume that $d(i, i) = 0$.

Consider a competitive facility location problem such that the set of facilities I and the set of clients J coincide with the vertex set V of a given graph $G(V, E)$. Suppose also that, for each $j \in J$, an order relation $\prec_j$ on I is determined by the lengths of the shortest paths to the vertex j. Assume that $i \prec_j k$ if $d(i, j) < d(k, j)$ or $d(i, j) = d(k, j)$ and $i < k$. Under the above assumptions our competitive facility location problems will be called *network competitive facility location problems*.

Let us show that if G is a path (i.e., it has the vertices of degree at most 2) then the problem (19)–(27) on G and the problem (19)–(21), (28)–(34) on G can be solved in polynomial time.

Fix a solution (x_i) of problems (19)–(27) and (19)–(21), (28)–(34). Let $I_0 = \{i \in I \mid x_i = 1\}$ be a set $\{i_1, i_2, \dots, i_K\}$ where $0 = i_0 < i_1 < i_2 < \dots < i_K < i_{K+1} = n + 1$. For convenience, we assume that I_0 also includes the fictive elements 0 and $n + 1$. Therefore, append the set of vertices V of G by 0 and $n + 1$ and assume that $f_{n+1} = 0$, $b_{n+1} = 0$, and $i \prec_j 0$, $i \prec_j n + 1$ for all $j \in J$ and $i \in I$.

Note that, by the properties of the relations $\prec_j$ on the vertices of the path, the feasible solutions $((z_i), (z_{ij}))$ of the inner problems (23)–(27) and (29)–(34) have the following properties for all $k = 1, \dots, K + 1$:

- if $z_i = 1$ and $i \in \{i_{k-1}, \dots, i_k\}$ then $z_{ij} = 0$ for every $j \notin \{i_{k-1}, \dots, i_k\}$;
- if $z_i = 1$ and $i \notin \{i_{k-1}, \dots, i_k\}$ then $z_{ij} = 0$ for every $j \in \{i_{k-1}, \dots, i_k\}$.

It follows that, for a fixed solution (x_i), each of the inner problems (23)–(27) and (29)–(34) splits into the $K+1$ inner problems such that kth inner problem, $k = 1, \dots, K+1$, differs from the corresponding original problem in that the sets I and J are replaced by I_k and J_k, $I_k = J_k = \{i_{k-1}, \dots, i_k\}$. Moreover, the kth inner problem is considered for a fixed vector (x_i), $i \in I_k$, where $x_{i_{k-1}} = x_{i_k} = 1$ and $x_i = 0$ for $i \neq i_{k-1}$, $i \neq i_k$.

Let $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ denote an optimal solution of the kth inner problem, $k = 1, \dots, K+1$. Note that, for $k = 1, \dots, K+1$, by the properties of the relations $\prec_j$ on the vertices of a path, for every optimal solution $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ in the case of the problem (23)–(27), the vector $(\widetilde{z}_i^k)$ will have at most two unit components; and, in the case of the problem (29)–(34), there exists an optimal solution $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ such that the vector $(\widetilde{z}_i^k)$ has at most two unit components. It follows that, for a fixed solution (x_i) in the case of either problems (19)–(27) and (19)–(21), (28)–(34), a desired optimal solution of the corresponding inner problem can be found in time $O(n^3)$.

Taking into account this for a fixed solution (x_i), we can represent the objective function (19) of the problem (19)–(27) as follows:

$$\min_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \Big\{ -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j \Big(1 - \sum_{i\in I} \widetilde{z}_{ij}\Big)\Big\}$$
$$= -\sum_{i\in I} f_i x_i + \sum_{j\in J} b_j - \max_{(\widetilde{z}_i),(\widetilde{z}_{ij})} \sum_{j\in J}\sum_{i\in I} b_j \widetilde{z}_{ij}$$
$$= \sum_{k=1}^{K+1} \Big\{ f_{i_k} + \sum_{j=i_{k-1}+1}^{i_k} b_j - \max_{(\widetilde{z}_i^k),(\widetilde{z}_{ij}^k)} \sum_{j=i_{k-1}+1}^{i_k} \sum_{i=i_{k-1}+1}^{i_k} b_j \widetilde{z}_{ij}^k \Big\}.$$

Set

$$l(i_{k-1}, i_k) = -f_{i_k} + \sum_{j=i_{k-1}+1}^{i_k} b_j - \max_{(\widetilde{z}_i^k),(\widetilde{z}_{ij}^k)} \sum_{j=i_{k-1}+1}^{i_k} \sum_{i=i_{k-1}+1}^{i_k} b_j \widetilde{z}_{ij}^k,$$

where $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ is an optimal solution of the kth inner problem of (23)–(27), $k = 1, \dots, K,$. Then we obtain the following equivalent formulation of the problem (19)–(27) on a path:

$$\max_{K} \max_{i_1,\dots,i_K} \sum_{k=1}^{K+1} l(i_{k-1}, i_k); \tag{41}$$

$$0 = i_0 < i_1 < \dots < i_K < i_{K+1} = n+1. \tag{42}$$

The problem (19)–(21), (28)–(34) on a path can be represented in the same form. For this problem, the function $l(i_{k-1}, i_k)$ is written as

$$l(i_{k-1}, i_k) = -f_{i_k} + \sum_{j=i_{k-1}+1}^{i_k} b_j - \sum_{j=i_{k-1}+1}^{i_k} \sum_{i=i_{k-1}+1}^{i_k} b_j \widetilde{z}_{ij}^k,$$

where $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ is an optimal solution of the kth inner problem of (29)–(34), $k = 1, \dots, K$. Note that, for the either problems, by the above mentioned properties of optimal solutions $\big((\widetilde{z}_i^k), (\widetilde{z}_{ij}^k)\big)$ of the inner problems, the calculation of $l(i_{k-1}, i_k)$ on every pair of points can be done in $O(n^3)$ time.

Given algorithm for solving the problem (41), (42) is based on the dynamic programming method [2, 3]. For every $i \in \{0, 1, \dots, n+1\}$, let $S(i)$ denote the optimal value of the objective function of the following problem:

$$\max_K \max_{i_1,\dots,i_K} \sum_{k=1}^{K+1} l(i_{k-1}, i_k); \qquad 0 = i_0 < i_1 < \dots < i_K < i_{K+1} = i.$$

Assume that $S(0) = 0$.

The algorithm for solving (41), (42) consists of the two phases: The first phase includes $n + 1$ steps. At the ith step, $i = 1, \dots, n+1$, $S(i)$ is computed by the formula

$$S(i) = \max_{0 \le i' < i} \{S(i') + l(i', i)\}.$$

The second phase consists of a preliminary step and finitely many main steps. At the preliminary step, put $i'_0 = n + 1$. At tth main step, $t = 1, 2, \dots$, the algorithm seeks for the element i'_t, $0 \le i'_t < i'_{t-1}$, such that

$$S(i'_{t-1}) = S(i'_t) + l(i'_t, i'_{t-1}).$$

If $i'_t > 0$ then the next step starts. If $i'_t = 0$ then put $K = t - 1$ and construct the optimal solution $\{i_1, \dots, i_K\}$, where $i_k = i'_{K-k+1}$, $k = 1, \dots, K$; and after the completion the algorithm terminates.

Provided that the function $l(i_{k-1}, i_k)$ is given, the algorithm for solving the problem (41), (42) runs in time $O(n^2)$ and computing all required values of $l(i_{k-1}, i_k)$ can be implemented in $O(n^5)$ time.

Thus, we proved the following

Theorem 1. *The competitive facility location problem* (19)–(27) *and* (19)–(21), (28)–(34) *on a path with n vertices is polynomial-time solvable with time complexity $O(n^5)$.*

4. UPPER BOUND ON THE OBJECTIVE FUNCTION OF COMPETITIVE FACILITY LOCATION PROBLEMS

Consider the problem (19)–(27) and its equivalent formulation as a pseudo-Boolean two-level programming problem (35)–(40). Construct a pseudo-Boolean function $f(y_i, \dots, y_m)$ such that, at every $(0, 1)$-vector $y = (y_i)$, the value of f does not exceed the value of the objective function (35). Call this pseudo-Boolean function $f(y_i, \dots, y_m)$ the *estimation* function.

To construct the estimation pseudo-Boolean function for every $j_0 \in J$ define the set $I_{j_0} \subset I$ as follows: Let $i_0 \in I$. Consider the set

$$N(i_0, j_0) = \{i \in I \mid i \prec_{j_0} i_0\}$$

and the set

$$J(i_0) = \{j \in J \mid i_0 \preccurlyeq_j i \text{ for every } i \notin N(i_0, j_0)\}$$

that gives the points $j \in J$ for which the element i_0 is more preferable than each $i \notin N(i_0, j_0)$. This set is nonempty because $j_0 \in J(i_0)$. If $N(i_0, j_0) = \varnothing$ then we assume by definition that $i_0 \in I_{j_0}$. If $N(i_0, j_0) \neq \varnothing$ then, for all $k \in N(i_0, j_0)$, construct

$$J(k, i_0) = \{j \in J(i_0) \mid k \prec_j i_0\}$$

that defines the points $j \in J(i_0)$ for which the element k is more preferable than i_0. We assume that $i_0 \in I_{j_0}$ if and only if, for every $k \in N(i_0, j_0)$,

$$g_k > \sum_{j \in J(k, i_0)} b_j.$$

Note that, for each $j_0 \in J$, the set I_{j_0} is nonempty because if $i_0 \preccurlyeq_{j_0} i$ for each $i \in I$ then $i_0 \in I_{j_0}$.

The following establishes an important property of I_{j_0}:

Lemma 1. *If* (x_i) *is a solution of the problem* (19)–(27) *such that* $x_i = 0$ *for all* $i \in I_{j_0}$ *then, for an optimal solution* $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ *of the problem* (23)–(27) *at which the objective function* (19) *takes the minimum value, we have*

$$\sum_{i \in I} \widetilde{z}_{ij_0} = 1.$$

Proof. Consider an optimal solution $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ of (23)–(27) satisfying the property in the assumptions of the lemma and assume that $\widetilde{z}_{ij_0} = 0$ for all $i \in I$. Let $i_0 \in I$ be such that $x_{i_0} = 1$ and $x_i = 0$, $\widetilde{z}_i = 0$ for all $i \in N(i_0, j_0)$. By the definition of $J(i_0)$, for any $j \in J(i_0)$, we have $i_0 \prec_j i$ for each $i \in I$ such that $\widetilde{z}_i = 1$. Since $i_0 \notin I_{j_0}$, there exists $k \in N(i_0, j_0)$ for which $J(k, i_0)$ has the property

$$g_k \leq \sum_{j \in J(k,i_0)} b_j.$$

Thus, we see that, for some $k \in I$, $\widetilde{z}_k = 0$, there exists a set $J(k, i_0)$ such that, for all $j \in J(k, i_0)$, we have $k \prec_j i_0$ and $i_0 \prec_j i$ for every $i \in I$, $\widetilde{z}_i = 1$. Furthermore, the above inequality holds for the set. This means that the solution $\big((\widetilde{z}'_i), (\widetilde{z}'_{ij})\big)$ differing from the original optimal solution $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ only in that $\widetilde{z}'_k = 1$ and $\widetilde{z}'_{kj} = 1$, $j \in J(i_0, k)$, also will be an optimal solution of the problem (23)–(27) and, moreover, will provide a smaller value of the objective function (19). This contradicts the choice of the optimal solution $((\widetilde{z}_i), (\widetilde{z}_{ij}))$ and proves Lemma 1. □

Consider the problem (35)–(40) and, by using the above-established, prove the following

Lemma 2. *For every solution* $y = (y_i)$ *of the problem* (35)–(40) *and an optimal solution* $\widetilde{u} = (\widetilde{u}_i)$ *of* (39), (40) *at which the objective function* (35) *takes the maximum value,*

$$\prod_{i \preccurlyeq i_{j_0}(\widetilde{u})} y_i \geq \prod_{i \in I_{j_0}} y_i \quad \text{for all} \quad j_0 \in J.$$

Proof. If $y_i = 0$ for some $i \in I_{j_0}$ then the inequality holds. Let $y_i = 1$ for every $i \in I_{j_0}$. Then by Lemma 1 for the optimal solution $(\widetilde{u}_i)$ of the problem (39), (40) on which the objective function (35) takes the maximum value, $i_{j_0}(\widetilde{u}) \prec i_{j_0}(y)$. Therefore, $y_i = 1$ for any $i \preccurlyeq i_{j_0}(\widetilde{u})$ and, consequently, the desired inequality holds. The proof of Lemma 2 is over. □

Consider a pseudo-Boolean function of the form

$$f^0(y_1, \dots, y_m) = -\sum_{i \in I} f_i y_i + \sum_{j \in J} b_j \prod_{i \in I_j} y_i.$$

By Lemma 2 for every vector $y = (y_i)$,

$$\max_{(\widetilde{u}_i)} \Big\{ -\sum_{i \in I} f_i y_i + \sum_{j \in J} b_j \prod_{i \preccurlyeq i_j(\widetilde{u})} y_i \Big\} \geq f^0(y_1, \dots, y_m).$$

Hence, $f^0(y_1, \dots, y_m)$ is an estimation function and

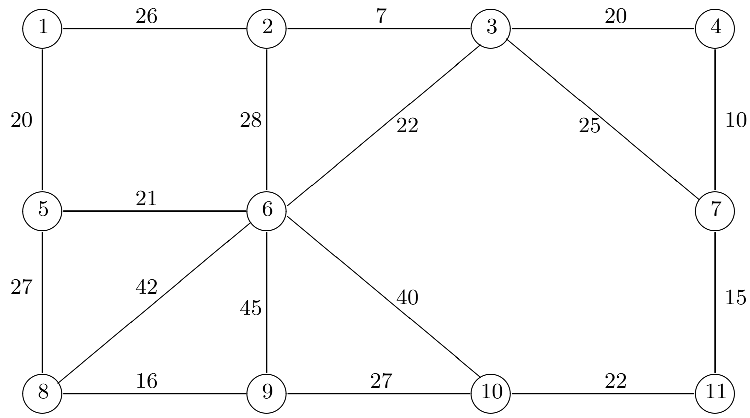
$$f^0 = \min_{(y_i)} \Big\{ f^0(y_1, \dots, y_m) + F \prod_{i \in I} y_i \Big\},$$

where $F > \min_{i \in I} f_i$, is a lower bound for the objective function (35) of the problem (35)–(40). Thus, we arrive at the following

Theorem 2. *The number*

$$UB = \sum_{j \in J} b_j - \sum_{i \in I} f_i - f^0$$

is an upper bound for the optimal value of the objective function (19) *of the problem* (19)–(27).



A network G with 11 vertices

The above upper bound, as was mentioned earlier, is an upper bound for the optimal value of the objective function (19) of (19)–(21), (28)–(34).

The algorithm for computing the upper bound for the objective function of the problems (19)–(27) and (19)–(21), (28)–(34) includes the two phases: In the first phase, the estimation pseudo-Boolean function is constructed and, in the second phase, its minimum value is computed. Constructing the estimation function reduces to finding I_j for each $j \in J$. From the definition of this set it follows that the set can be constructed in time $O(m^2n^2)$. For solving the minimization problem for the pseudo-Boolean function equivalent to the uncapacitated facility location problem [3] we can use quite a few algorithms [3] based on the ideas of implicit enumeration and local search.

5. A NUMERICAL EXAMPLE AND RESULTS OF A COMPUTATIONAL EXPERIMENT

We present the results of the algorithm constructing the upper bound for an instance of the problem (19)–(27) on the network G depicted in the figure.

The network has 11 vertices with the numbers indicated in the centers of the circles representing the vertices of the network. The distances between the neighboring vertices defining for the clients the facility selection rules are indicated near the edges connecting the neighboring vertices. The input data of the instance are given in the table.

The income b_j realized from client j, the fixed costs f_i and g_i of opening facility i by the leader firm and the follower firm

	1	2	3	4	5	6	7	8	9	10	11
b_j	10	15	20	10	15	20	10	15	20	5	5
f_i	28	28	28	28	28	28	28	28	28	28	28
g_i	18	18	18	18	18	18	18	18	18	18	18

The optimal solution of the instance is the vector $(x_i^\star)$, where $x_3^\star = x_9^\star = 1$, $x_i^\star = 0$ for $i \neq 3$, $i \neq 9$; and the corresponding optimal solution of the inner problem is defined by the vector $(\widetilde{z}_i^\star)$, where $\widetilde{z}_4^\star = \widetilde{z}_5^\star = 1$, $\widetilde{z}_i^\star = 0$ for $i \neq 4$, $i \neq 5$. The facilities of the leader firm will serve the clients located at the vertices with numbers 2, 3, 8, 9, and 10 delivering the total income equal to 75. Therefore, the maximum profit of the leader firm is equal to 19.

The sets $I_j,\ j = 1, \dots, 11$, required for computing the upper bound in the case of the instance are as follows:

$$\begin{array}{lll} I_1 = \{1,2,5\}, & I_5 = \{1,5,6,8\}, & I_9 = \{9\}, \\ I_2 = \{1,2,3\}, & I_6 = \{6\}, & I_{10} = \{6,7,9,10,11\}, \\ I_3 = \{3\}, & I_7 = \{4,7,10,11\}, & I_{11} = \{4,7,9,10,11\}. \\ I_4 = \{3,4,7\}, & I_8 = \{5,6,8,9\}, & \end{array}$$

The minimum value f^0 of the corresponding estimation function attains at the vector $(y_i^\star)$ and is equal to -207, where $y_3^\star = y_6^\star = 0$, $y_i^\star = 1$ for $i \neq 3$ and $i \neq 6$. Therefore, the upper bound is

$$\sum_{j=1}^{11} b_j - \sum_{i=1}^{11} f_i - f^0 = 145 - 308 + 207 = 44,$$

while the optimal value of the objective function is 19.

To show the efficiency of the suggested upper bound we give the results of a computational experiment consisting in computing an upper bound for the objective function of the problem (19)–(27) on a path and comparing them with the optimal values of the objective function.

Computations were done for two classes of the problems with input data produced as follows:

- The number of the vertices is 100.
- The distance between neighboring vertices is a uniformly distributed random variable with integer values between 1 and 6.
- The income b_j realized from client j is a uniformly distributed random variable with integer values between 5 and 15.
- The fixed costs f_i of opening facility i by the leader firm is a constant value equal to 40.
- The fixed costs of opening facility i by the follower firm is a constant value equal to 35 for the first class of the problems and 40 for the second.

Computations were done for the series of 20 problems from each class. For the specified problem, we computed the accuracy of the upper bound equal to the ratio of the value of the upper bound to the optimal value of the objective function. For the first class of problems, the average value of the accuracy is equal to 1.81 and, for the second class, 1.47.

The retrieved values of the accuracy seem to be quite acceptable and leave hope for the efficient utilization of the above-suggested upper bound when constructing branch-and-bound type algorithms for solving the competitive facility location problems. Notice also that the optimal solutions of the competitive facility location problem (pseudo-Boolean two-level programming problems) resulted from computations and the optimal solutions of the minimization problem for the corresponding estimation pseudo-Boolean functions have the sets of zero components close with respect to the number and the structure. This leaves hope for the utilization of the optimal solution of the minimization problem for the estimation function as a base for constructing approximation solutions for the competitive facility location problems studied in this paper.

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