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New Lower Bounds for the Facility Location Problem with Clients' Preferences

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Abstract—A bilevel facility location problem in which the clients choose suppliers based on their own preferences is studied. It is shown that the coopertative and anticooperative statements can be reduced to a particular case in which every client has a linear preference order on the set of facilities to be opened. For this case, various reductions of the bilevel problem to integer linear programs are considered. A new statement of the problem is proposed that is based on a family of valid inequalities that are related to the problem on a pair of matrices and the set packing problem. It is shown that this formulation is stronger than the other known formulations from the viewpoint of the linear relaxation and the integrality gap.

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INTRODUCTION

In facility location problems with clients' preferences (see [1, 2]), there are two levels of decision making. At the upper level, a set of facilities to be opened is chosen. Then, at the lower level, the clients are assigned to these facilities according to the clients' preferences. The problem is to choose the facilities to be opened so as to serve all the clients with minimizing the total cost.

First, the location problems with clients' preferences were considered in [3]. Later, similar models were independently proposed in [2, 4]. If the clients' preferences at the lower level are in agreement with the matrix of the transportation cost at the upper level, we have the classical facility location problem (see [1]). Therefore, the problems with clients' preferences (the p-median problem, the simple facility location problem, and their generalizations) are NP-hard in the strong sense and do not belong to the class APX (see [5]). In [1, 2], the close relationships of these problems with pseudo-Boolean functions was established. It was shown that these problems are equivalent; more precisely, given a location problem with clients' preferences, an equivalent minimization problem for a pseudo-Boolean function can be constructed in a polynomial time and conversely. In [6, 7], it was shown how this property of the location problem can be used to reduce its dimension.

In [8] the problem with a fixed number of facilities was studied. A genetic algorithm was proposed for finding near optimal solutions. This algorithm uses local optima under Lin–Kernighan neighborhoods as individuals of the population. The proposed approach was tested on instances with a considerable integrality gap and showed good performance.

To find an exact solution, the problem is reduced to an integer linear program (ILP). In [9] well-known and new valid inequalities were considered. They help improve lower bounds and the efficiency of the branch-and-cut method. In [6, 8], various formulations of the problem in terms of integer linear programming were considered, and a formulation based on reducing the original problem to the problem on a pair of matrices was proposed. Using this formulation, an improved lower bound can be obtained by increasing the number of variables.

In this paper, we propose a new formulation based on the analysis of the reduction to the problem on a pair of matrix (see [6, 8]). It provides a lower bound that is not worse than that obtained in [6], but this bound is obtained using a new family of valid inequalities rather than by increasing the number of variables. The relationship of this problem with the set packing problem is examined. The valid inequalities for this problem can be used to solve the facility location problem with clients' preferences. It is shown that the inequalities proposed in [9] are a particular case of these valid inequalities.

The paper is organized as follows. In Section 1, the problem statement is presented and its properties are analyzed. In Section 2, a review of the known ILP formulations is given and a new family of valid inequalities is proposed. The relationship with the set packing problem is investigated in Section 3. In the final section, we discuss the results and the lines of further studies.

1. STATEMENT OF THE PROBLEM

Introduce the following notations. $I = \{1, 2, \dots, m\}$ is the set of facilities; $J = \{1, 2, \dots, n\}$ is the set of clients; $f_i \geq 0$ ($i \in I$) is the cost of opening the facility i; $c_{ij} \geq 0$ ($i \in I, j \in J$) is the matrix of the production and delivery costs for servicing the clients; $g_{ij} \geq 0$ ($i \in I, j \in J$) is the matrix of the clients' preferences; more precisely, if $g_{i_1 j} < g_{i_2 j}$, then the client j prefers the open facility i_1 to the open facility i_2.

The variables of the problem are as follows:

$$y_i = \begin{cases} 1, & \text{if the facility } i \text{ is opened,} \\ 0, & \text{otherwise,} \end{cases}$$

$$x_{ij} = \begin{cases} 1, & \text{if the client } j \text{ is served by facility } i, \\ 0, & \text{otherwise.} \end{cases}$$

Using this notation, we obtain the following bilevel programming problem (see [4, 10]): find

$$\min_y \sum_{i \in I} \sum_{j \in J} c_{ij} x^*_{ij}(y) + \sum_{i \in I} f_i y_i \tag{1}$$

subject to

$$y_i \in \{0, 1\}, \quad i \in I, \tag{2}$$

where $x^*_{ij}(y)$ is the optimal solution of the client problem: find

$$\min_x \sum_{i \in I} \sum_{j \in J} g_{ij} x_{ij} \tag{3}$$

subject to

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J, \tag{4}$$

$$x_{ij} \leq y_i, \quad i \in I, \quad j \in J, \tag{5}$$

$$x_{ij} \in \{0, 1\}, \quad i \in I, \quad j \in J. \tag{6}$$

Objective function (1) of the upper level problem gives the cost of servicing the clients and opening the facilities. Objective function (3) of the lower level problem guarantees that the clients are served in conformity with their preferences. Constraints (4) ensure that each client is served by exactly one facility. Inequalities (5) indicate that the clients can be served only by open facilities.

Problem (1)–(6) is denoted by BLP, and its objective function (1) is denoted by $F(y, x^*(y))$. This function is interpreted as the total expenses of the first decision maker (DM_1) that makes decisions at the upper level. For the given vector y, the solution $x^*(y)$ is the optimal choice of suppliers in accordance with the clients' preferences. In the general case, this choice is not unique. Then, the BLP needs to be refined; namely, a more precise definition of the optimal solution must be given. In particular, one can consider cooperative and anticooperative strategies for the DM_1 and the second decision maker DM_2 who makes decisions at the lower level. If the DM_2 wants to minimize (maximize) the total expenses of the DM_1, we have a cooperative (anticooperative) statement of the problem. Below, we consider a simpler case when a unique optimal decision of the DM_2 exists for any decision of the DM_1. This property is guaranteed if the entries of the matrix g_{ij} in every column $j \in J$ are distinct. In other words, for any pair of facilities, every client prefers one of them. In this case, the objective function $F(y, x^*(y))$ is uniquely determined by the vector y, and we may write $F(y)$ instead of $F(y, x^*(y))$. Thus, by the optimal solution of the problem, we mean the vector y^* satisfying constraints (2) and minimizing $F(y)$.

Let us show that the cooperative and anticooperative statements of the problem can be reduced to this particular case. Denote by Opt(y) the set of the optimal solutions of problem (3)–(6) for the given vector y. Then, the BLP in the cooperative statement can be written as

$$\min_{y,\, x \in \mathrm{Opt}(y)} \{F(y, x) | y_i \in \{0, 1\}, i \in I\}. \tag{7}$$

In the anticooperative statement, it is written as

$$\min_{y} \max_{x \in \mathrm{Opt}(y)} \{F(y, x) | y_i \in \{0, 1\}, i \in I\}. \tag{8}$$

Theorem 1. *Problems* (7) *and* (8) *can be reduced to the BLP with the unique clients' optimal choice.*

Proof. Consider problem (7). Given its initial data, we construct a new equivalent BLP problem in which the entries in every column of the preference matrix g'_{ij} are distinct. The new problem differs from the original one only in this matrix.

For every column $j \in J$ of the matrix (g_{ij}) in problem (7), we define a permutation $\pi(j) = (\pi_1, \ldots, \pi_m)$ of the elements of the set I such that

$$g_{\pi_1 j} \le \ldots \le g_{\pi_m j}.$$

If $g_{\pi_i j} = g_{\pi_{i+1} j}$, we assume that $c_{\pi_i j} \le c_{\pi_{i+1} j}$. Set $g'_{\pi_i j} = i$ for all $i \in I$. By construction, all the entries of the matrix (g'_{ij}) in every column are different. Therefore, for any y, the set Opt(y) consists of a single element. It is easy to verify that the optimal solution of the new problem is also optimal for problem (7). The case of the anticooperative setting is considered in a similar manner. The theorem is proved.

Assume that the strategy of the DM_2 is not known. For example, we may assume that he chooses an element in Opt(y) depending on y. In this case, problems (7) and (8) give a lower and an upper bounds of the optimum in this difficult-to-formalize problem.

It is known (see [2, 8]) that the BLP with the unique clients' optimal choice can be reduced to an ILP. There are several reduction techniques that differ in the number of variables and constraints. Define the sets $S_{ij} = \{k \in I | g_{kj} < g_{ij}\}$ and $T_{ij} = \{k \in I | g_{kj} > g_{ij}\}$ $(i \in I, j \in J)$. Note that we have the following implication for the optimal solution of the client problem:

$$(x_{ij} = 1) \Rightarrow (y_k = 0), \quad k \in S_{ij}. \tag{9}$$

Using this property, the BLP can be written as follows (see [2, 6]): find

$$\min \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i \tag{10}$$

subject to

$$y_k + x_{ij} \le 1, \quad k \in S_{ij}, \quad i \in I, \quad j \in J, \tag{11}$$

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J, \tag{12}$$

$$0 \le x_{ij} \le y_i, \quad i \in I, \quad j \in J, \tag{13}$$

$$x_{ij}, y_i \in \{0, 1\}, \quad i \in I, \quad j \in J. \tag{14}$$

Indeed, for the optimal solution of problem (10)–(14), all the constraints of the original problem are satisfied, and constraints (11) ensure that x_{ij} is an optimal solution of the client problem. We also can consider the bilevel problem with a fixed number of facilities to be opened, this is a so-called p-median problem (see [8]). In this case, the following constraint is added to the integrality constraints at the upper level:

$$\sum_{i \in I} y_i = p; \tag{15}$$

here, $p \in Z_+$ is the number of facilities to be opened. For this problem, all the assertions that were discussed for the BLP are valid, as well as a similar reduction to an ILP with additional constraint (15). In the next section, we propose techniques for strengthening formulation (10)–(14).

2. STRENGTHENING THE FORMULATION

For many ILPs, there exist several equivalent formulations. The quality of a formulation is usually estimated by the integrality gap defined as gap = (Opt – LP)/Opt, where Opt is the optimal value and LP is the value of the linear relaxation. The less the integrality gap, the stronger is the formulation. A perfect formulation exactly describes the convex hull of the set of feasible integer points. However, the derivation of such a formulation is equivalent to solving the original problem (see [11]). In many cases, this is practically impossible. For formulation (10)–(14), the integrality gap can be as large as 20–30% (see [7, 9]). Even in the particular case $f_i = 0$ for $i \in I$, this value can be arbitrarily close to 100% (see [6, 7]). Below, we propose a technique for strengthening the original formulation of the problem so as to reduce the integrality gap. Note that this approach does not rule out Gomory cuts or any other cuts for solving problems (e.g., see [11–13]).

Definition 1. Let U be a set of points in $\mathbb{R}^n$. The inequality $a^{\mathrm{T}}u \le b$ is said to be *valid* for U if $a^{\mathrm{T}}u \le b$ for all $u \in U$.

Denote by P_c the polyhedron of problem (10)–(14), which is the convex hull of the integer points satisfying constraints (11)–(14). Let LB_1 be the optimum in linear programing problem (10)–(13).

2.1. Well-known Valid Inequalities

For the polyhedron P_c, several valid inequalities are known. They induce various formulations, which differ in the number of variables, the number of constraints, and, as a result, in the integrality gap.

1. *Single client preference inequalities* (see [3]) are written as

$$C1(i, j): \quad y_i + \sum_{k \in T_{ij}} x_{kj} \le 1, \quad i \in I, \quad j \in J. \tag{16}$$

If the facility i is open, then the client j is not served by less preferable facilities, i.e., by the facilities belonging to the set T_{ij}. These inequalities dominate inequalities (11). The lower bound given by the linear program (10), (12), (13), subject to (16) is denoted by LB_2. We have $LB_1 \le LB_2$.

2. In [9], a strengthening of inequalities (16) was proposed. Let $j_1, j_2 \in J$ and $i \in I$. If the facility i is open, the client j_1 will not use the facilities from the set T_{ij_1} and the client j_2 will not use the facilities from T_{ij_2}; that is,

$$C2(i, j_1, j_2): \quad \sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{k \in T_{ij_2} \cap S_{ij_1}} x_{kj_2} + y_i \le 1. \tag{17}$$

We call these inequalities *preferences of a pair of clients.*

3. Inequalities (17) can be extended to the case of an arbitrary number of clients. Let $j_1, \ldots, j_s \in J$ and $i \in I$. Then, the inequalities

$$Cs(i, j_1, \ldots, j_s): \sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{t=2}^{s} \sum_{k \in T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right)} x_{kj_t} + y_i \le 1, \tag{18}$$

are valid for P_c (see [9]). They induce an exponential number of additional constraints. Some of them can dominate the others. For that reason, only a part of these inequalities should be used. In [9], it is proposed to choose the elements $j_1, \ldots, j_s \in J$ for which the sets T_{ij_t} $(t = 1, 2, \ldots, s)$ are mutually disjoined.

4. Also, inequalities dominating (13) were proposed in [9]. Let $j_1, j_2 \in J$ and $i \in I$. Then,

$$\text{if} \quad S_{ij_2} \subseteq S_{ij_1}, \quad \text{then} \quad x_{ij_1} \le x_{ij_2}. \tag{19}$$

For $S_{ij_1} = S_{ij_2}$, we have $x_{ij_1} = x_{ij_2}$.

Denote by LB_4 the optimum in linear program (10), (12), (13), (19) with all the inequalities (18). By LB_3, we denote the optimum in linear programing problem (10), (12), (13), (19) with the subset of inequalities (18) proposed in [9]. Then, $LB_2 \le LB_3 \le LB_4$.

2.2. Reduction to the Problem on a Pair of Matrices

Consider the matrices $A = (a_{ij})$ $(i \in I, j \in J_1)$ and $B = (b_{ij})$ $(i \in I, j \in J_2)$ containing the same number of rows. The problem on a pair of matrices (see [1]) is to find a nonempty subset $S \subseteq I$ that provides the minimum for the objective function

$$\sum_{j \in J_1} \max_{i \in S} a_{ij} + \sum_{j \in J_2} \min_{i \in S} b_{ij}.$$

If A is a diagonal matrix, we obtain the simple facility location problem. In [2, 4], a reduction of the BLP to the problem on a pair of matrices was proposed. On the basis of this reduction, a new formulation of the original problem in terms of ILP was obtained in [8, 6].

Let us represent the matrix (c_{ij}) as the sum of two matrices $c_{ij} = a_{ij} + b_{ij}$. For each $j \in J$, given the matrix (g_{ij}), we find the permutation $\pi(j)$ and set

$$a_{\pi_1 j} = 0, \quad b_{\pi_1 j} = c_{\pi_1 j},$$

$$a_{\pi_k j} = \sum_{l=1}^{k-1} \min\{0, c_{\pi_{l+1} j} - c_{\pi_l j}\}, \quad k = 2, 3, \ldots, m,$$

$$b_{\pi_k j} = c_{\pi_1 j} + \sum_{l=1}^{k-1} \max\{0, c_{\pi_{l+1} j} - c_{\pi_l j}\}, \quad k = 2, 3, \ldots, m.$$

Let $\Delta_l^j = \min\{0; c_{\pi_{l+1} j} - c_{\pi_l j}\}$ $(l = 1, 2, \ldots, m-1)$ and $L_j = \{l \in \{1, 2, \ldots, m-1\} | \Delta_l^j < 0\}$. Note that, for the given $j \in J$, we can uniquely determine $\pi_l \in I$ by $l \in L_j$. For each $j \in J$, we define the nonnegative matrix

$$\bar{a}_{il} = \begin{cases} 0, & \text{if} \quad i \in T_{\pi_l j}, \\ -\Delta_l^j, & \text{if} \quad i \notin T_{\pi_l j}, \end{cases} \quad i \in I, \quad l \in L_j.$$

By the construction of the matrix $(\bar{a}_{il})$, we have

$$a_{ij} = \sum_{l \in L_j} (\bar{a}_{il} + \Delta_l^j), \quad i \in I, \quad j \in J.$$

Now, the BLP can be written as

$$\min_{y_i \in \{0,1\}} \left\{ \sum_{j \in J} \sum_{l \in L_j} \max_{i | y_i = 1} \bar{a}_{il} + \sum_{j \in J} \min_{i | y_i = 1} b_{ij} + \sum_{i | y_i = 1} f_i \right\} + \sum_{j \in J} \sum_{l \in L_j} \Delta_l^j. \tag{20}$$

Define the auxiliary variables $v_l^j \in \{0, 1\}$ ($l \in L_j, j \in J$) and rewrite this problem as follows (see [8]): find

$$\min\left\{\sum_{j \in J}\sum_{l \in L_j} -\Delta_l^j v_l^j + \sum_{i \in I}\sum_{j \in J} b_{ij}x_{ij} + \sum_{i \in I} f_i y_i\right\} + \sum_{j \in J}\sum_{l \in L_j} \Delta_l^j \tag{21}$$

subject to

$$y_i + \sum_{k \in T_{ij}} x_{kj} \le 1, \quad i \in I, \quad j \in J, \tag{22}$$

$$\sum_{i \in I} x_{ij} = 1, \quad j \in J, \tag{23}$$

$$0 \le x_{ij} \le y_i, \quad i \in I, \quad j \in J, \tag{24}$$

$$v_l^{j_1} \ge \sum_{i \notin T_{\pi_l j_1}} x_{ij_2}, \quad l \in L_j, \quad j_1 \in J, \quad j_2 \in J, \tag{25}$$

$$v_l^j, y_i, x_{ij} \in \{0, 1\}, \quad l \in L_j, \quad j \in J, \quad i \in I. \tag{26}$$

Denote by LB_5 the optimum in linear program (21)–(25). It can be proved (see [8]) that $LB_5 \ge LB_2$.

In fact, the reduction to the problem on a pair of matrices is another method for obtaining lower bounds by constructing extended formulations. Indeed, the resulting formulation is a problem in the larger space of variables $(x, y, v) \in \mathbb{B}^{m \cdot n} \times \mathbb{B}^m \times \mathbb{B}^{|L_1| + \ldots + |L_n|}$. The original formulation involves the space of variables $(x, y) \in \mathbb{B}^{m \cdot n} \times \mathbb{B}^m$. A clear drawback of the extended formulations is the increase in the number of variables; the attempts to strengthen the formulation in the original space usually result in a large number of additional constraints (for example, in inequalities (18)). A possible way of preventing the excessive growth of the extended formulations is using them as the basis for the construction of new valid inequalities and the corresponding separation algorithms [11, 13].

Theorem 2. *The inequalities*

$$\sum_{i \in T_{\pi_l j_1}} x_{ij_1} + \sum_{i \notin T_{\pi_l j_1}} x_{ij_2} \le 1, \quad l \in L_j, \quad i \in I, \quad j_1, j_2 \in J, \quad j_1 \ne j_2, \tag{27}$$

are valid for P_c.

Proof. Let $(\hat{x}, \hat{y}, \hat{v})$ be an optimal solution of problem (21)–(26).

1. First, we prove that inequality (25) turns into an equality for $j_1 = j_2$ on the optimal solution; that is,

$$\hat{v}_l^j = \sum_{i \notin T_{\pi_l j}} \hat{x}_{ij}, \quad l \in L_j, \quad j \in J. \tag{28}$$

Assume the converse, and let $\hat{v}_l^j = \sum_{i \notin T_{\pi_l j}} \hat{x}_{ij} + s_l^j$ for $l \in L_j$ and $j \in J$, where $s_l^j \ge 0$ for $l \in L_j$, and $\sum_{j=1}^n \sum_{l \in L_j} s_l^j > 0$. Then, we have

$$\sum_{i=1}^m \sum_{j=1}^n c_{ij}\hat{x}_{ij} + \sum_{i=1}^m f_i \hat{y}_i = \sum_{j \in J}\sum_{l \in L_j} \Delta_l^j (1 - \hat{v}_l^j) + \sum_{i \in I}\sum_{j \in J} b_{ij}\hat{x}_{ij} + \sum_{i=1}^m f_i \hat{y}_i$$

$$= \sum_{j \in J}\sum_{l \in L_j} \Delta_l^j \left(\sum_{i \in T_{\pi_l j}} \hat{x}_{ij} - s_l^j\right) + \sum_{i \in I}\sum_{j \in J} b_{ij}\hat{x}_{ij} + \sum_{i=1}^m f_i \hat{y}_i$$

$$= \sum_{j \in J} \sum_{k=2}^{m} \hat{x}_{\pi_k j} \sum_{l=1}^{k-1} \Delta_l^j + \sum_{i \in I} \sum_{j \in J} b_{ij} \hat{x}_{ij} + \sum_{i=1}^{m} f_i \hat{y}_i - \sum_{j \in J} \sum_{l \in L_j} \Delta_l^j s_l^j$$

$$= \sum_{j \in J} \sum_{i \in I} a_{ij} \hat{x}_{ij} + \sum_{i \in I} \sum_{j \in J} b_{ij} \hat{x}_{ij} + \sum_{i=1}^{m} f_i \hat{y}_i - \sum_{j \in J} \sum_{l \in L_j} \Delta_l^j s_l^j$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \hat{x}_{ij} + \sum_{i=1}^{m} f_i \hat{y}_i - \sum_{j \in J} \sum_{l \in L_j} \Delta_l^j s_l^j > \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \hat{x}_{ij} + \sum_{i=1}^{m} f_i \hat{y}_i.$$

We arrived at a contradiction.

2. Replacing (25) by

$$v_l^{j_1} = \sum_{i \notin T_{\pi_l j_1}} x_{ij_1}, \quad l \in L_j, \quad j_1 \in J, \tag{29}$$

$$v_l^{j_1} \geq \sum_{i \notin T_{\pi_l j_1}} x_{ij_2}, \quad l \in L_j, \quad j_1, j_2 \in J, \quad j_1 \neq j_2, \tag{30}$$

we obtain an equivalent problem. Substituting (29) into (30), we obtain the original problem with additional inequalities

$$\sum_{i \notin T_{\pi_l j_1}} x_{ij_1} \geq \sum_{i \notin T_{\pi_l j_1}} x_{ij_2}, \quad l \in L_j, \quad j_1, j_2 \in J, \quad j_1 \neq j_2,$$

which is equivalent to formulation (21)–(26). Due to (23), we conclude that inequalities (27) are valid for P_c, which completes the proof.

Denote by LB_6 the optimal value in the linear programming relaxation of problem (10), (16), (12)–(14) subject to additional inequalities (27).

Corollary 1. It holds that $LB_5 \leq LB_6$.

Thus, returning to the space of the original variables, we constructed new valid inequalities. The resulting lower bound LB_6 is not worse than the known bound LB_5. In the next section, we describe one more method for constructing valid inequalities based directly on the analysis of the original formulation.

3. CLIQUE INEQUALITIES

Now, we show the relationship of the BLP with the well-known set packing problem. The properties of this problem's polyhedron are well studied (see [12]). They can be used to obtain families of effective cutting planes in problems with a similar structure (see [14–17]).

Consider a 0–1 matrix D and a nonnegative vector d. The *set packing* problem with the variables z is formulated as follows:

$$\max_z \{ d^{\mathrm{T}} z : Dz \leq 1 \}.$$

It is equivalent to finding a maximum weighted independent set in the graph $G = (V, E)$ constructed as follows. Every column of D is associated with a vertex in G. The vertices i and j are connected by an edge if and only if the columns i and j are not orthogonal.

Denote by P_G the polyhedron of the set packing problem, which is the convex hull of the 0–1 vectors corresponding to the independent sets of the graph G. Consider one class of valid inequalities for P_G that we will need below. Any complete subgraph in a given graph is called *clique*. A clique that is not a part of a larger clique is called a *locally maximal clique.* Let K be a clique in G. It is known (see [12]) that the *clique inequality*

$$\sum_{k \in K} z_k \le 1$$

is valid for P_G and it is facet defining if K is a locally maximal clique. In the formulation of the location problem, a group of inequalities can be distinguished that define the relaxation of this problem to a set packing problem. Therefore, we can use the available families of valid inequalities for P_G to solve the facility location problem.

Return to the inequalities $C2(i, j_1, j_2)$ $(i \in I, j_1, j_2 \in J)$:

$$\sum_{i \in T_{ij_1}} x_{kj_1} + \sum_{k \in T_{ij_2} \cap S_{ij_1}} x_{kj_2} + y_i \le 1.$$

Consider the relaxation of the problem under examination to the set packing problem defined by this family of inequalities. We construct a family of clique inequalities that are also valid for P_c. Now, add these inequalities to the LP relaxation of problem (10), (12)–(14), (19). The corresponding lower bound is denoted by LB_7.

Theorem 3. *It holds that* $LB_4 \le LB_7$.

Proof. It is sufficient to show that inequalities (18) are clique inequalities for the proposed relaxation to the set packing problem. For simplicity, we denote the graph vertices by the indexes of the corresponding variables; that is, the vertex (i, j) corresponds to the variable x_{ij}, and the vertex i corresponds to the variable y_i. By $W(i, j)$, we denote the set of vertices (i, j) the variables of which appear in the sum $\Sigma_{k \in T_{ij}} x_{kj}$; by $W(i, j_1, j_2)$, we denote the vertices corresponding to the sum

$$\sum_{k \in T_{ij_2} \cap S_{ij_1}} x_{kj_2};$$

and by $WS(i, j_t)$, we denote the vertices corresponding to the sum

$$\sum_{k \in T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right)} x_{kj_t}.$$

We prove the theorem by induction on the parameter s.

Induction hypothesis, $s = 1$.The inequalities $Cs(i, j_1)$ coincide with inequalities (16). Since they are dominated by the inequalities $C2(i, j_1, j_2)$ $(j_2 \in J)$, which were used to construct the graph, the vertices $\{i\} \cup W(i, j)$ form a clique. For $s = 2$, the inequalities $Cs(i, j_1, j_2)$ coincide with $C2(i, j_1, j_2)$. Therefore, the vertices $\{i\} \cup W(i, j_1) \cup W(i, j_1, j_2)$ also form a clique.

Induction step. Assume that

$$\sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{t=2}^{s} \sum_{k \in T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right)} x_{kj_t} + y_i \le 1, \quad i \in I, \quad j_1, \ldots, j_s \in J,$$

are clique inequalities for a certain s; that is, we assume that the vertices from the set

$$\{i\} \cup W(i, j_1) \cup \left(\bigcup_{t=2}^{s} WS(i, j_t)\right)$$

form a clique. Consider the inequalities $C(s+1)(i, j_1, \ldots, j_{s+1})$:

$$\sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{t=2}^{s+1} \sum_{k \in T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right)} x_{kj_t} + y_i$$

$$= \sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{t=2}^{s} \sum_{k \in T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right)} x_{kj_t} + y_i + \sum_{k \in T_{ij_{s+1}} \cap \left(\bigcap_{q=1}^{s} S_{ij_q}\right)} x_{kj_{s+1}} \le 1.$$

We want to prove that the vertices from the set

$$\{i\} \cup W(i, j_1) \cup \left(\bigcup_{t=2}^{s} WS(i, j_t)\right)$$

and the vertices from the set $WS(i, j_{s+1})$ form a clique. Indeed, for any $t = 1, 2, \ldots, s$, the inequalities $C2(i, j_t, j_{s+1})$

$$\sum_{k \in T_{ij_t}} x_{kj_t} + \sum_{k \in T_{ij_{s+1}} \cap S_{ij_t}} x_{kj_{s+1}} + y_i \le 1, \quad i \in I, \quad j_t \in J,$$

were used in the construction of the graph. Therefore, the sets of vertices

$$\{i\} \cup W(i, j_t) \cup W(i, j_t, j_{s+1}), \quad t = 1, 2, \ldots, s,$$

form cliques. Consider two cases.

(i) For $t = 1$, we have the inequality

$$\sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{k \in T_{ij_{s+1}} \cap S_{ij_1}} x_{kj_{s+1}} + y_i \le 1, \quad i \in I.$$

Since

$$\bigcap_{q=1}^{s} S_{ij_q} \subseteq S_{ij_1},$$

we have

$$WS(i, j_{s+1}) \subseteq W(i, j_1, j_{s+1}).$$

Therefore, the vertices of the sets $\{i\}$, $W(i, j_1)$, $WS(i, j_{s+1})$ are mutually connected by edges.

(ii) Let $t = 2, 3, \ldots, s$. Since

$$T_{ij_t} \cap \left(\bigcap_{q=1}^{t-1} S_{ij_q}\right) \subseteq T_{ij_t},$$

we have

$$WS(i, j_t) \subseteq W(i, j_t).$$

Taking into account the fact that $\bigcap_{q=1}^{s} S_{ij_q} \subseteq S_{ij_t}$, we conclude that the vertices of the sets $WS(i, j_t)$ and $WS(i, j_{s+1})$ are also mutually connected by edges, which completes the proof.

Note that inequalities (18) have the structure of the polyhedron of the set packing problem. However, in the proof of Theorem 3, we actually proved, in addition to the inequality $LB_4 \le LB_7$, that no new edges can be produced using inequalities (18) when the graph of the set packing problem is constructed. Therefore, there is no need to consider an exponentially large number of constraints, when the problem is relaxed to the polyhedron of the set packing problem.

As has already been mentioned above, the use of the inequalities $C2(i, j_1, j_2)$ ($i \in I, j_1, j_2 \in J$) is not the only way to construct the graph. Introduce the additional variables $y_i' = 1 - y_i$. Then, inequalities (13) can be written in the form $x_{ij} + y_i' \le 1$. Now, the following group of constraints can be written for the problem under examination:

$$y_i + \sum_{k \in T_{ij}} x_{kj} \le 1, \quad i \in I, \quad j \in J,$$

$$\sum_{i \in I} x_{ij} \le 1, \quad j \in J,$$

$$\sum_{k \in T_{ij_1}} x_{kj_1} + \sum_{k \in T_{ij_2} \cap S_{ij_1}} x_{kj_2} + y_i \le 1, \quad i \in I, \quad j_1, j_2 \in J, \tag{31}$$

$$x_{ij} + y_i' \le 1, \quad i \in I, \quad j \in J,$$

$$\sum_{i \in T_{\pi_l j_1}} x_{ij_1} + \sum_{i \notin T_{\pi_l j_1}} x_{ij_2} \le 1, \quad l \in L_j, \quad i \in I, \quad j_1, j_2 \in J, \quad j_1 \ne j_2.$$

The set packing problem defined by inequalities (31) also is a relaxation of the original problem. Denote by LB_8 the lower bound obtained by the LP relaxation of problem (10), (12)– (14), (19) with the clique inequalities for the new system (31). Then, we have $LB_8 \ge LB_7$ and $LB_8 \ge LB_6$.

CONCLUSIONS

In this paper, we investigated the lower bounds for the facility location problem with clients' preferences. Some known equivalent formulations of the ILP for this problem were considered and new formulations were proposed that differ in the integrality gap. The results can be represented by the scheme

$$LB_1 \le LB_2 \le \begin{matrix} LB_3 \le LB_4 \le LB_7 \\ LB_5 \le LB_6 \end{matrix} \le LB_8.$$

The lower bound LB_8 based on a new family of valid inequalities and on the cuttings of the set packing problem dominates the other known lower bounds and opens new possibilities for developing exact methods.

The bounds LB_7 and LB_8 were obtained using clique inequalities. Other families of valid inequalities for the set packing problem are also known, for example, odd-hole inequalities (see [12]) and others (see [18–20]). They can also be used for improving the lower bounds. To implement this approach, efficient algorithms for finding the desired inequalities are needed. There can be an exponentially large number of such inequalities, which makes the problem difficult. The development of efficient algorithms for finding the appropriate inequalities is an important direction of further research.

More general location models (for example, competitive and dynamic models in which the clients' preferences are taken into account, see [10, 21]) are also of interest. It seems likely that all the location models can be generalized for the case when the clients' preferences are explicitly taken into account. The study of such models and techniques for solving the corresponding optimization problems is the subject for further research.

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