Lecture 8

Competitive Facility Location

Let us assume that two firms want to open facilities. The first firm, which we will refer to as the ***leader***, opens its own set of facilities X from the set I. We assume that $|X| = p$. Later, the second firm, which we will refer to as the ***follower***, opens its own set of facilities Y from the set $I \setminus X$. We assume that $|Y| = r$. Each user selects one facility from the union $X \cup Y$ according to its own preferences, for example, according to distances to the facilities. Each firm will get a positive profit w_j if it services the user j. The firms try to maximize own profits. They do not have the same rights. The leader makes a decision first. The follower makes a decision by analyzing the set X. It is a Stakelberg game for two players, where we need to maximize the total profit of the leader.

Leader-Follower Facility Location Game

Given: I is the set of facilities;

J is the set of clients;

w_j is the profit from client j;

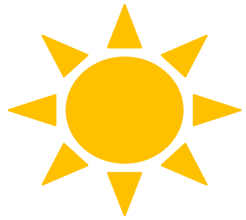
p is the number of leader facilities;

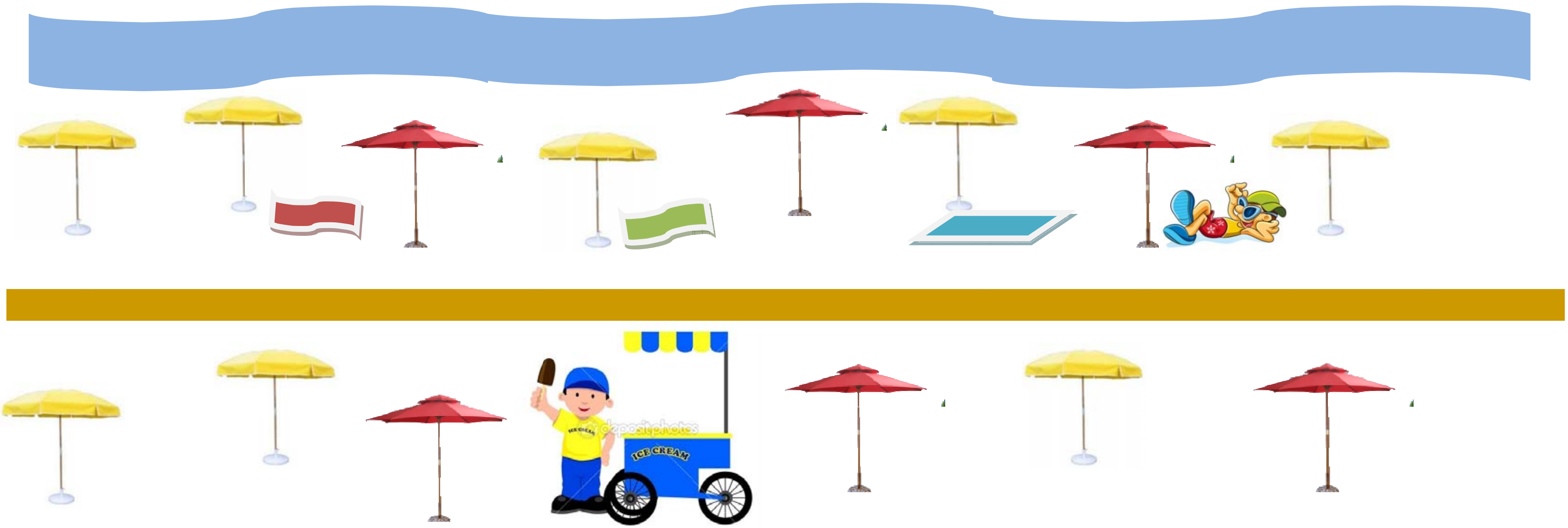
r is the number of follower facilities;

d_{ij} is the distance between facility i and client j.

Find: maximal profit of the leader taking into account the best strong reaction of the follower.

Example. Ice cream in the beach,





Hometask

In the Euclidean plane we given n clients with coordinates (x_j, y_j) and demand w_j, $j = 1, \ldots, n$. Two players, a leader and a follower open an own facility to capture the clients. At first, the leader opens a facility in arbitrary point at the plane. Later on, the follower opens own facility against the leader's one. Each client patronizes the closes facility. In case of ties, the leader's facility is preferred. Each player tries to maximize his market share. The goal of the game is to find a point for the leader facility to maximize his market share. Design a polynomial time algorithm for the game with optimal strategy for the leader.

Decision Variables

$$x_i = \begin{cases} 1, & \text{if the leader opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if the follower opens facility } i \\ 0 & \text{otherwise} \end{cases}$$

$$z_j = \begin{cases} 1, & \text{if client } j \text{ is served by a leader facility} \\ 0, & \text{if client } j \text{ is served by a follower facility} \end{cases}$$

For each vector x and each client j we can define the set of facilities

$$I_j(x) = \{i \in I \mid g_{ij} < \min_{l \in I} \left(g_{lj} \,\middle|\, x_l = 1\right)\},$$

which allow "capturing" client j by the follower. If a client has the same distanc-es to the closest leader and the closest follower facilities, he prefers the leader facility.

Mathematical Model

$$\max_{x} \sum_{j \in J} w_j z_j^*(x)$$

s.t.

$$\sum_{i \in I} x_i = p,$$

$$x_i \in \{0,1\}, \;\; i \in I, \;\; z_j^*(x) \in \mathcal{F}(x),$$

where $\mathcal{F}(x)$ is the set of optimal solutions of the follower problem:

$$\max_{z,y} \sum_{j \in J} w_j (1 - z_j)$$

s.t.

$$1 - z_j \le \sum_{i \in I_j(x)} y_i \, , \quad j \in J;$$

$$\sum_{i \in I} y_i = r;$$

$$x_i + y_i \le 1, \;\; y_i, z_j \in \{0,1\}, \quad i \in I, j \in J.$$

Can the leader get a half of the market, if $p = r$ *?*

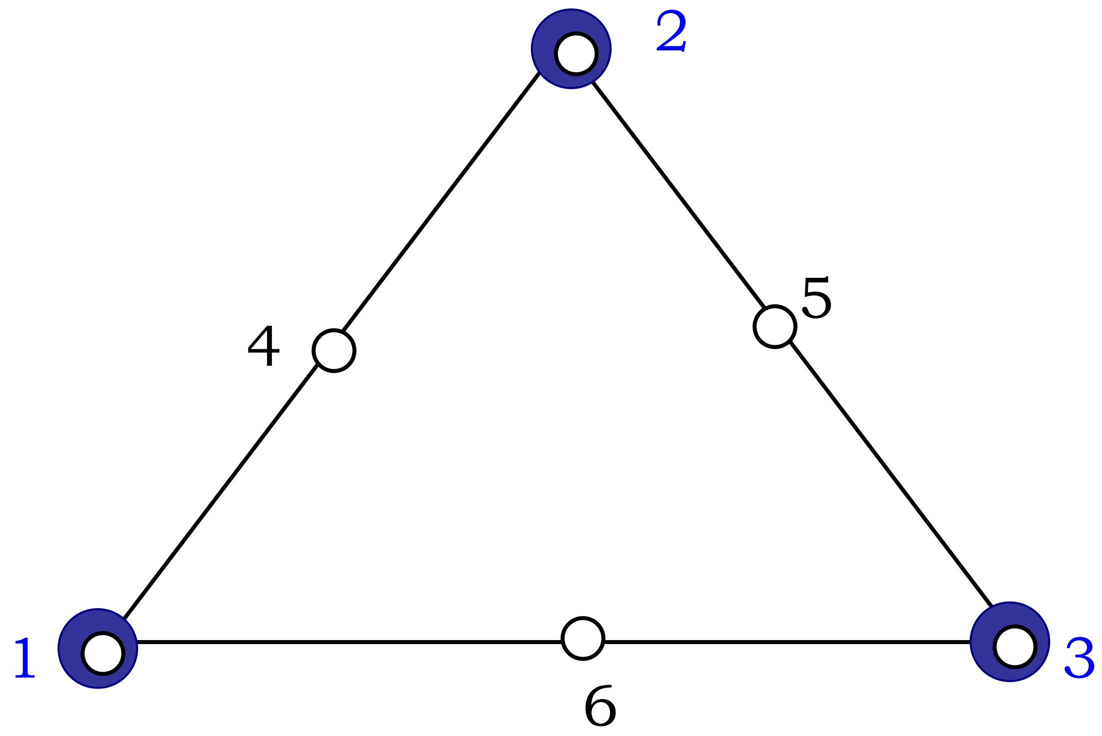
Hopeless example

$I \ = \ \{1, 2, 3, 4, 5, 6\};$

$J \ = \ \{1, 2, 3\};$

$w_j = 1, \ j \in J;$

$p = r = 1.$



Variants of the Model

- Conservative and nonconservative clients
- The French rule: follower cannot open facility very close to the leader facility
- The gravity rule: each client visits all open facilities and divides own demand inverse proportionally to the distances
- The covering rule: each facility has a service region
- Design of each facility: ⋆, ⋆⋆, … , ⋆⋆⋆⋆⋆ hotels
- Budget constraints, profit maximization, elastic demands.

Well-posed and ill-posed problems.

Singe Level Reformulation

Denote by $\mathcal{F}$ the set of all feasible solutions for the follower.

For $y \in \mathcal{F}, j \in J$, we define the set of facilities

$$I_j(y) = \{i \in I \mid d_{ij} \le \min_{l \in I}(d_{lj} \mid y_l = 1)\},$$

which allow to the leader to get client j against the follower solution y.

Now we introduce additional variables:

W is total profit of the leader

$$t_{ij} = \begin{cases} 1, & \text{if facility } i \text{ of the leader is closest to client } j \\ 0 & \text{otherwise.} \end{cases}$$

Large Scale Reformulation

$$\max W$$

s.t.

$$\sum_{j \in J} \sum_{i \in I_j(y)} w_j t_{ij} \geq W, \quad y \in \mathcal{F};$$

$$\sum_{i \in I} t_{ij} = 1, \quad j \in J;$$

$$x_i \geq t_{ij}, \quad i \in I, j \in J;$$

$$\sum_{i \in I} x_i = p;$$

$$x_i, \quad t_{ij} \in \{0,1\}, \quad i \in I, j \in J.$$

Is it equivalent reformulation or not?

Let $F \subset \mathcal{F}$ and we replace $\mathcal{F}$ by F. What can we say about $W(F)$?

Exact Method

1. Select an initial set $F \subset \mathcal{F}$ and put $W^*:=0$.
2. Solve the problem for the set F and compute upper bound $W(F)$ and $x(F)$.
3. Solve the follower problem and compute $y(x(F))$ and lower bound $W(x,y)$.
4. If $W^* < W(x,y)$ then $W^*:=W(x,y)$.
5. If $W^* = W(F)$ then STOP.
6. Include $y(x(F))$ into the set F and go to 2.

Is it finite method?

Leader Market Share

$|I| = |J| = 50$