Completability of hamiltonian cycle in halved cube

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Hypercube and halved cube

$W(v) = \sum_{i=1}^{n} |v_i|$ — ***weight*** of $v \in \{0,1\}^n$.
$d(u,v) = W(u-v)$ — ***Hamming distance*** between u and v.

Hypercube Q_n: $V(Q_n) = \{0,1\}^n$; $E(Q_n) = \{(u,v) : d(u,v) = 1\}$.

Halved cube $\frac{1}{2}Q_n$: $V\left(\frac{1}{2}Q_n\right) = \{v : v \in \{0,1\}^n, W(v)\text{— odd}\}$;
$E\left(\frac{1}{2}Q_n\right) = \{(u,v) : d(u,v) = 2\}$.

Properties of $\frac{1}{2}Q_n$

- $|V\left(\frac{1}{2}Q_n\right)| = 2^{n-1}$;
- $|E\left(\frac{1}{2}Q_n\right)| = \binom{n}{2}2^{n-2}$;
- $\deg v = \binom{n}{2}$;
- $\frac{1}{2}Q_n = Q_{n-1}^2$;
- $\frac{1}{2}Q_n$ is hamiltonian.

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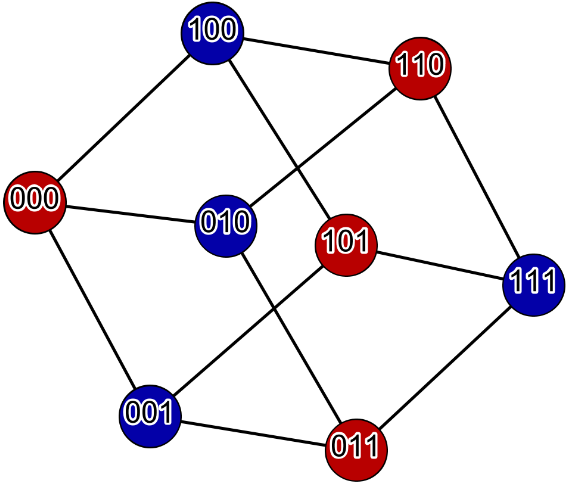

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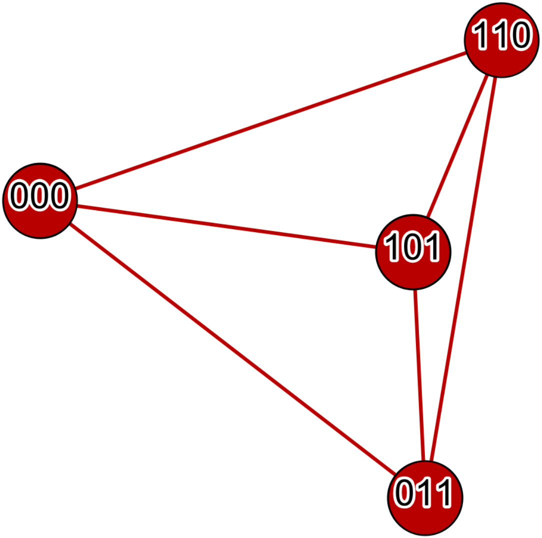

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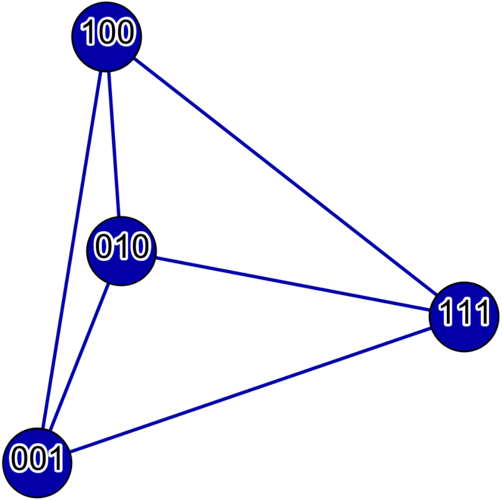

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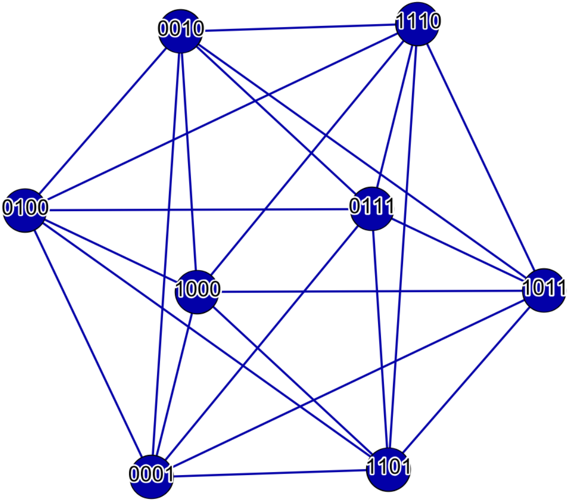

Halved cube $\frac{1}{2}Q_n$

W. Imrich, S. Klavzar and A. Veksel. A characterization of halved cubes (1998)

Let $n \geq 5$. Let G be a connected, $\binom{n}{2}$-regular graph on 2^{n-1} vertices. Then G is a halved cube $\frac{1}{2}Q_n$ if and only if

- every edge of G is contained in exactly two n-cliques.
- for any n-cliques K and K', $|K \cap K'| \neq 1$.

Completable hamiltonian cycle

Hamiltonian cycle $C = v_0, v_1, \ldots, v_{2^{n-1}-1}$ in Q_n ***generates*** hamiltonian cycle

$$v_1, v_3, v_5, \ldots, v_{2^{n-1}-1},$$

in $\frac{1}{2}Q_n$.

Hamiltonian cycle $C = v_0, v_1, \ldots, v_{2^{n-1}-1}$ in $\frac{1}{2}Q_n$ is called ***completable***, if there exist n-binary vectors $u_0, u_1, \ldots, u_{2^{n-1}-1}$ such that cycle

$$v_0, \boldsymbol{u_0}, v_1, \boldsymbol{u_1}, \ldots, v_{2^{n-1}-1}, \boldsymbol{u_{2^{n-1}-1}}$$

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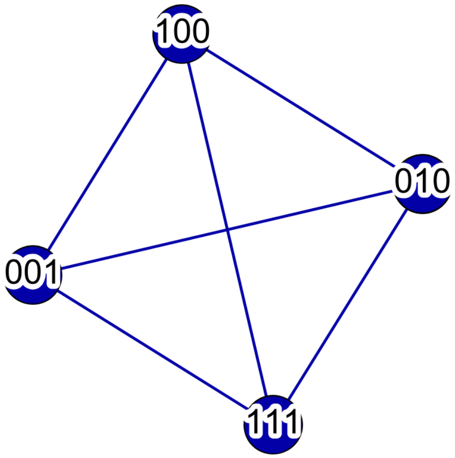

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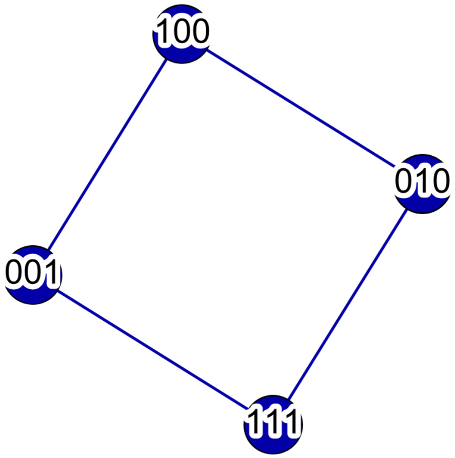

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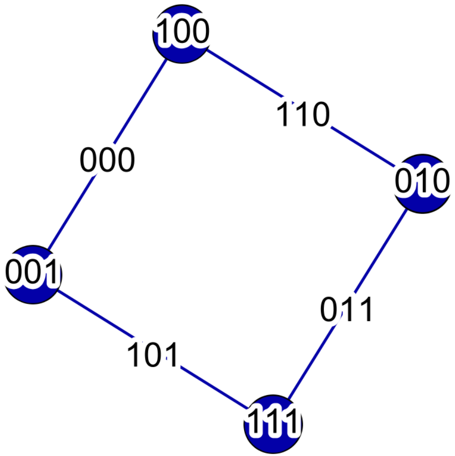

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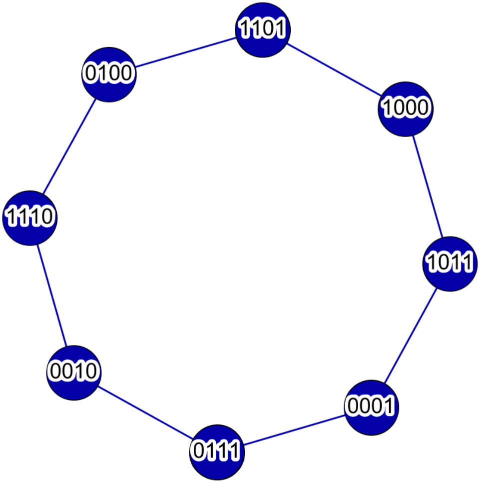

Completable hamiltonian cycle

Lemma 1

If for some even binary vector v hamiltonian cycle does not contain edge (u, w) such, that $d(u, v) = d(v, w) = 1$, then C is not completable.

Theorem 1

For any $n \geq 4$ there exists hamiltonian cycle in $\frac{1}{2}Q_n$, which is not completable.

Vertex v is called ***closest*** to edge (u, w) of $\frac{1}{2}Q_n$, if $d(u, v) = d(v, w) = 1$.

For every edge in $\frac{1}{2}Q_n$ there are only two closest vertices.

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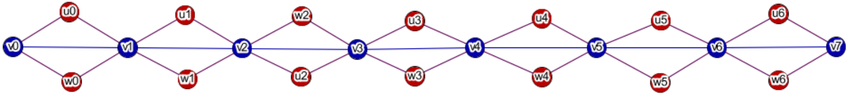

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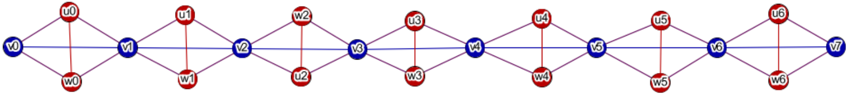

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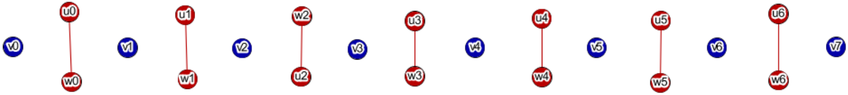

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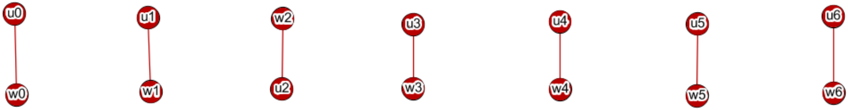
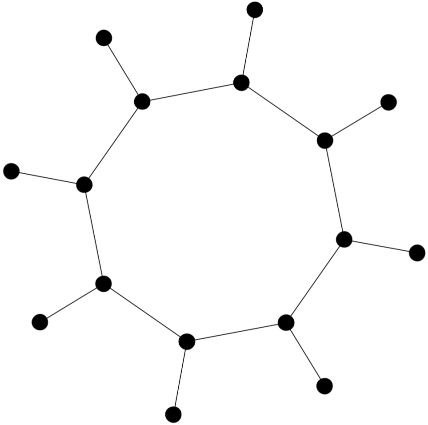


Properties of $\mathcal{G}(C)$

- $V(\mathcal{G}(C))$ — even n-binary words;
- $|V(\mathcal{G}(C))| = |E(\mathcal{G}(C))| = 2^{n-1}$;
- $\mathcal{G}(C)$ does not contain triangles;
- degree of any vertex in $\mathcal{G}(C)$ is not greater than $n-1$.

Examples of $\mathcal{G}(C)$ for $\frac{1}{2}Q_5$



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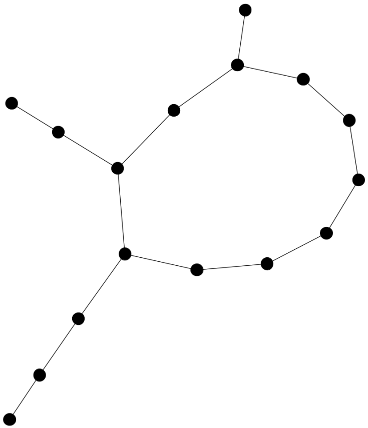

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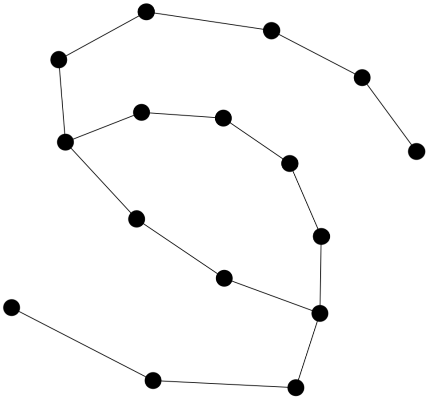

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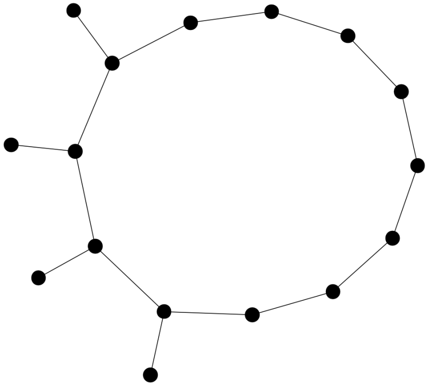

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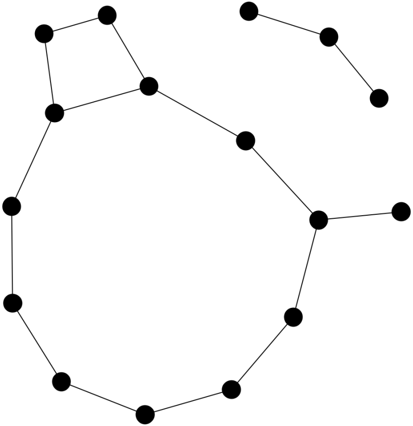

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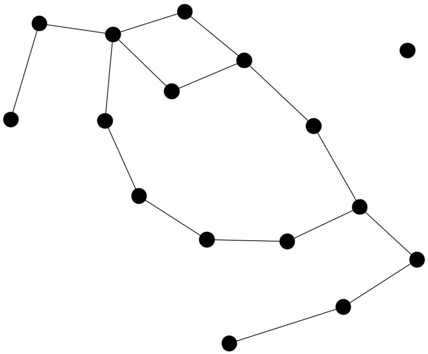

Completability of hamiltonian cycle in $\frac{1}{2}Q_n$

Theorem 2

Hamiltonian cycle C in $\frac{1}{2}Q_n$ is completable if and only if there is no trees among connected components of graph $\mathcal{G}(C)$.

There should be a bijection $\psi : V(\mathcal{G}(C)) \to E(\mathcal{G}(C))$, such that v and $\psi(v)$ are incident.

Lemma 2

For any connected graph $G = (V, E)$ with $|V| = |E|$, there is bijection $\psi : V \to E$ such, that $\forall w \in V$ vertex w and edge $\psi(w)$ are incident.

Corollary

If hamiltonian cycle C in $\frac{1}{2}Q_n$ is completable, then there is exactly 2^k ways to complete it (k — number of connected components $\mathcal{G}(C)$).

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n-dimensional Gray code

***n*-dimensional Gray code** is a cyclic list of all 2^n binary vectors of length n such, that two consecutive vectors differ in exactly one position.

Transition sequence of code is a cyclic word $(m_1, m_2, \ldots, m_{2^n})$ over alphabet $\{1, 2, \ldots, n\}$ such, that i-th and $(i+1)$-th vectors in differ in position m_i. Denote set of all transition sequences for n-dimensional Gray codes as Π_n.

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Transition sequence:
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Locally balanced Gray code

$l_1(C)$ — maximal length such, that in any subword of this length in transition sequence all elements are different.

$$l_1(n) = \max_{C \in \Pi_n} l_1(C)$$

Trivial estimates

For any $n \geq 3$:

$$2 \leq l_1(n) \leq n - 1.$$

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Locally balanced Gray codes. Survey

Evdokimov A., **On numeration of subsets of finite sets** (1980).

$$\frac{n}{2} \leq l_1(n) \leq n - 1.$$

Goddyn L., Lawrence G.M., Nemeth E. **Gray Codes with Optimized Run Lengths** (1988).

$$\frac{2}{3}n \leq l_1(n) \leq n - 1.$$

Goddyn L., Gvozdjak P., **Binary Gray Codes with Long Bit Runs** (2003).

$$n - \lceil 2.001 \log n \rceil \leq l_1(n) \leq n - 1.$$

For even n, n-Gray code with $l_1(C) = n - 1$ exists if and only if...

then there exists hamiltonian cycle in $\frac{1}{2}Q_n$

$$C_{1/2} = v_0, v_1, v_2, \ldots, v_{2^{n-1}-1},$$

such, that

- it is completable;
- vertices v_i and v_{i+n-1} are adjacent for any i;
- cycle $C'_{1/2} = \overline{v_0}, \overline{v_{n-1}}, \overline{v_{2(n-1)}}, \ldots$ is also completable;

$$f : E(C_{1/2}) \to E(C'_{1/2});$$

$$f(v_i, v_{i+1}) = (\overline{v_{i-\lfloor \frac{n-1}{2} \rfloor}}, \overline{v_{i+1+\lfloor \frac{n-1}{2} \rfloor}})$$

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