Eigensupports of minimal cardinality

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Sobolev Institute of Mathematics

The First Russian–Japanese Workshop on Algebraic combinatorics
29-30 August, 2016

Introduction: whom are we talking about?

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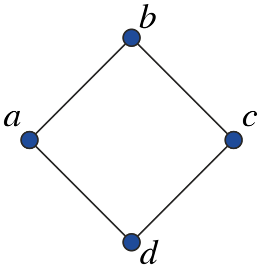

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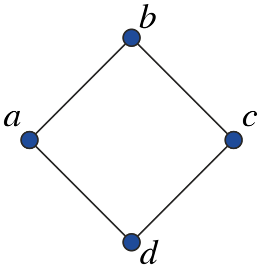

–Function on vertices–:

$$f = (f_a, f_b, f_c, f_d).$$

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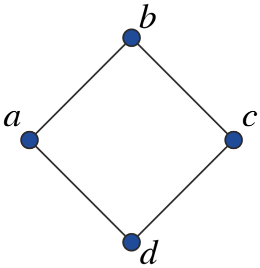

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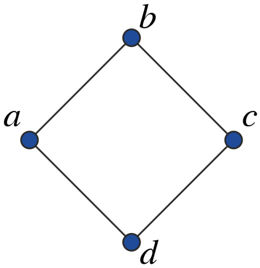

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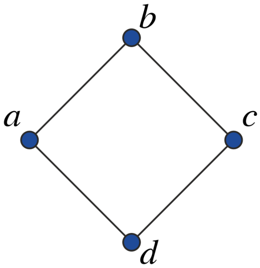

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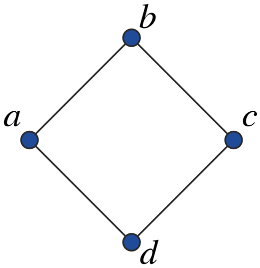

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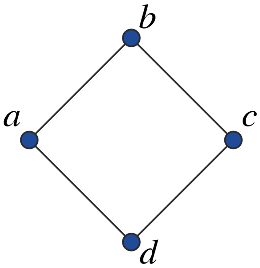

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$$\Lambda = \{2, 0, 0, -2\}$$

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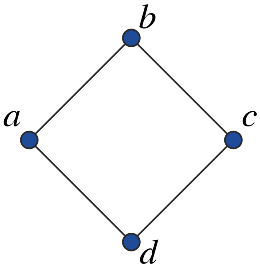

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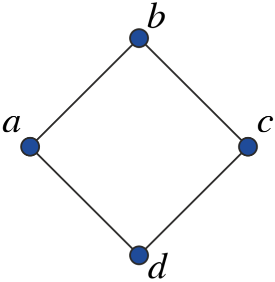

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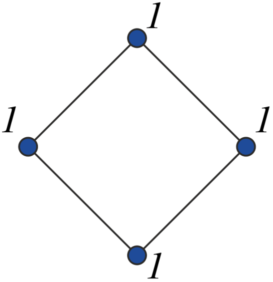

–**Eigenfunctions**–

$$\lambda = 2$$
$$f = (1, 1, 1, 1)$$

–**Function on vertices**–:

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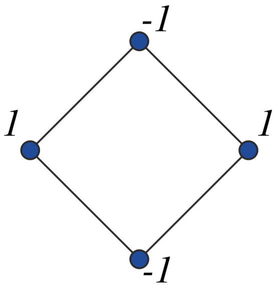

Eigenfunctions

$$\lambda = -2$$
$$f = (1, -1, 1, -1)$$

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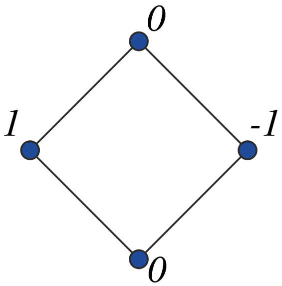

–**Eigenfunctions**–

$$\lambda = 0$$
$$f = (1, 0, -1, 0)$$

–**Function on vertices**–:

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Eigensupport

–Local definition–

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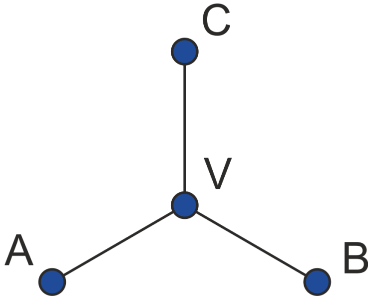

Eigensupport

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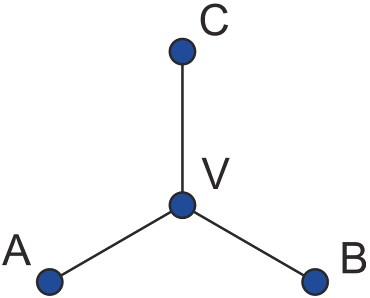

$$\lambda f_v = f_a + f_b + f_c.$$

Example:

Example: Cocktail Party Graph

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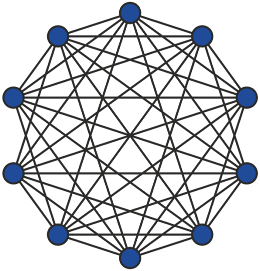

$t = 5$

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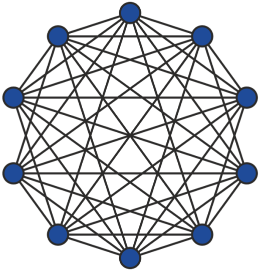

$t = 5$

Strongly Regular Graph:

$$\mathrm{srg}(n = 2t, n-2, n-4, n-2)$$

$$\Lambda = \{n-2, 0^t, -2^{t-1}\}$$

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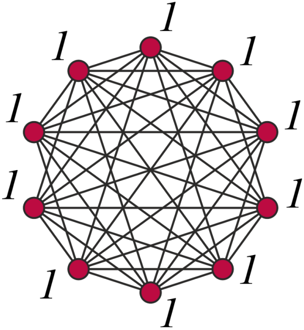
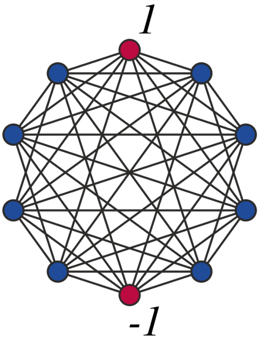


$\lambda = 8$

Cocktail Party Graph for $t = 5$:

$$\Lambda = \{8, 0^5, -2^4\}$$

Example: Cocktail Party Graph



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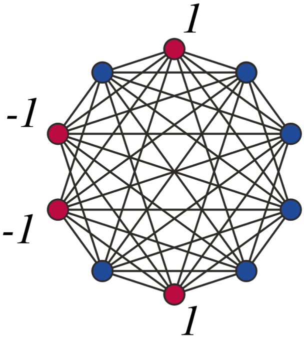

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So, what is our problem?

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Question: For a given graph G what are the **minimal** cardinalities of its eigensupports $\mathcal{S}_\lambda$?

Matrix Rank-distance graphs

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Let M_i be the matrix of size $n \times m$ over the finite field F_q. Define $\mathcal{M}$ to be the set of all such matrices M_i.

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So we obtain an undirected graph with q^{nm} vertices, which appears to be *distance-regular*.

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Intersection array: $\{9, 4; 1, 6\}$

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Intersection array: $\{9,4;1,6\}$
Eigenvalues: $\Lambda=\{9,1^9,-3^6\}$

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MRD Graphs: Let's start from something simple

$K_{3,3}$ — minimal eigensupport for $\lambda = -3$

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- ....

Thank you for your attention!

有難う 御座います