

Minimum supports of eigenfunctions of Hamming graphs

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$$\Sigma_q = \{0, 1, \dots, q - 1\}.$$

Definition

The **Hamming distance** $d(x, y)$ between vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ from Σ_q^n is the number of positions i such that $x_i \neq y_i$.

Definition

The **Hamming graph** $H(n, q)$ is a graph whose vertex set is Σ_q^n and two vertices are adjacent if the Hamming distance between them equals 1.

$G = (V, E)$ — is a simple graph.

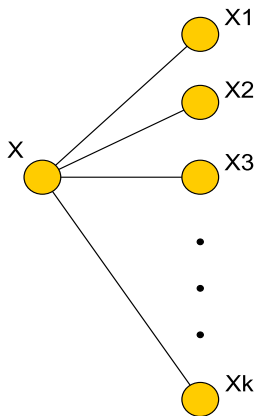
Definition

A function $f : V \rightarrow \mathbb{R}$ is called an *eigenfunction* of the graph G corresponding to an eigenvalue λ if

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

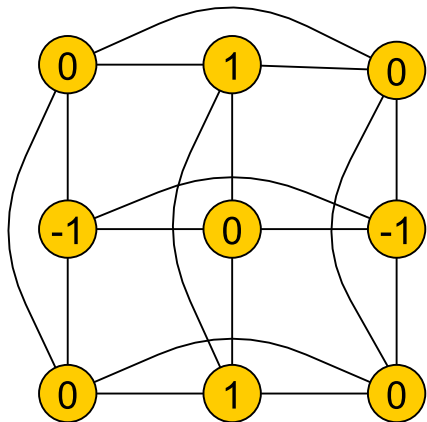
for any vertex x .

Basic definitions

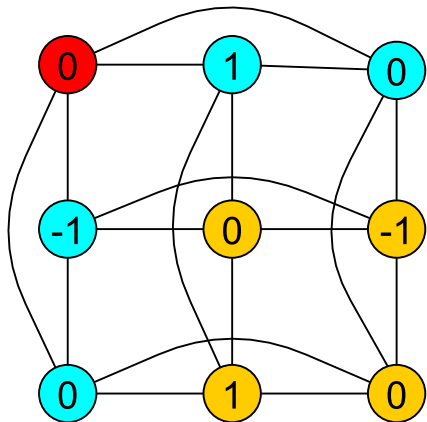


$$\lambda \cdot f(x) = f(x_1) + f(x_2) + \dots + f(x_k)$$

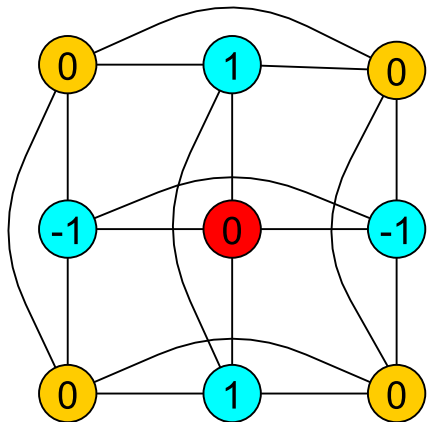
Example



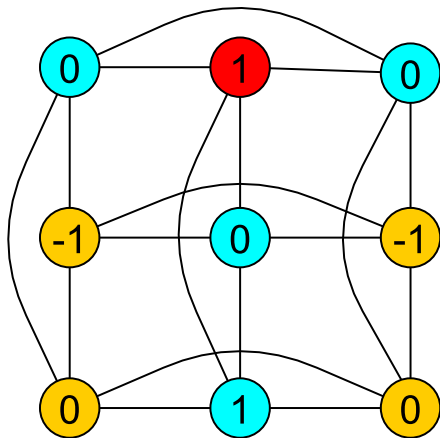
Example



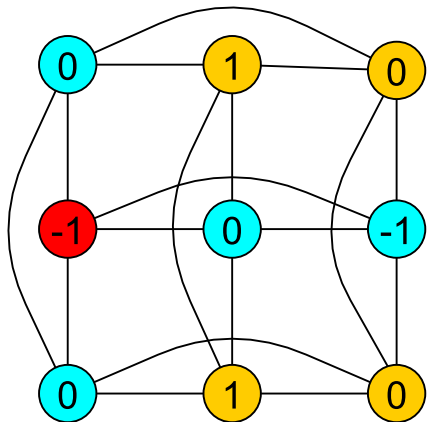
Example



Example



Example



Minimum support

Definition

The set $S(f) = \{x \in V \mid f(x) \neq 0\}$ is called the **support** of f .

Problem

To find the minimum cardinality of the support of eigenfunctions in the Hamming graphs.

Example

Example.

$x = (x_1, x_2)$ is a vertex of $H(2, q)$.

Let $f(x) = 1$ for $x_1 = i$ and $x_2 \neq j$;

$f(x) = -1$ for $x_2 = j$ and $x_1 \neq i$;

and $f(x) = 0$ otherwise.

Then f is an eigenfunction with $\lambda_1 = q - 2$.

Example

The set of vertices $x = (x_1, x_2, \dots, x_n)$ of $H(n, q)$ such that $x_i = k$ is denoted by $T_k(i, n)$.

Example. Let

$$f(x) = \begin{cases} 1, & \text{for } x \in T_k(i, n) \setminus T_m(j, n); \\ -1, & \text{for } x \in T_m(j, n) \setminus T_k(i, n); \\ 0, & \text{otherwise.} \end{cases}$$

Then f is an eigenfunction corresponding to $\lambda_1 = n(q-1) - q$ in $H(n, q)$.

The spectrum of $H(n, q)$ is
 $\{\lambda_m = n(q-1) - qm \mid m = 0, 1, \dots, n\}$.

Theorem (Potapov, 2012)

Let $f : H(n, q) \rightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_m and $f \not\equiv 0$. Then the following statements are true:

- $|S(f)| \geq 2^m$.
- If $q = 2$, then the minimum cardinality of the support f equals $\max(2^m, 2^{n-m})$.

Theorem (Vorob'ev, Krotov, 2014)

Let $f : H(n, q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_m and $f \not\equiv 0$. Then

$$|S(f)| \geq 2^m (q - 2)^{n-m}$$

for $\frac{mq^2}{2n(q-1)} > 2$ and

$$|S(f)| \geq q^n \left(\frac{1}{q-1}\right)^{m/2} \left(\frac{m}{n-m}\right)^{m/2} \left(1 - \frac{m}{n}\right)^{n/2}$$

for $\frac{mq^2}{2n(q-1)} \leq 2$.

Main theorem

The set of vertices $x = (x_1, x_2, \dots, x_n)$ of $H(n, q)$ such that $x_i = k$ is denoted by $T_k(i, n)$.

Theorem (V., 2016)

Let $f : H(n, q) \rightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_1 , $f \not\equiv 0$ and $q > 2$. Then $|S(f)| \geq 2(q-1)q^{n-2}$. Moreover, if $|S(f)| = 2(q-1)q^{n-2}$, then

$$f(x) = \begin{cases} c, & \text{for } x \in T_k(i, n) \setminus T_m(j, n); \\ -c, & \text{for } x \in T_m(j, n) \setminus T_k(i, n); \\ 0, & \text{otherwise.} \end{cases}$$

where $c \neq 0$ is a constant, i, j, k, m are some numbers and $i \neq j$.

Example

Example.

(x, y, z) is a vertex of $H(3, q)$.

Let $f(x) = 1$ for $(0, 0, z)$ with $z \neq 0$ and $(q, y, 0)$ with $y \neq 0$;

$f(x) = -1$ for $(0, y, 0)$ with $y \neq 0$ and $(q, 0, z)$ with $z \neq 0$;

and $f(x) = 0$ otherwise.

Then f is an eigenfunction with $\lambda_2 = q - 3$.

Theorem (V., 2016)

Let $f : H(3, q) \rightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_2 , $f \not\equiv 0$ and $q > 4$. Then $|S(f)| \geq 4(q - 1)$.

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