2-Arc-Transitive Covers

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Graphs and Groups, Spectra and Symmetries

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1. Coverings of a graph

A Covering **from a graph** X **to a graph** Y **means:**

$\exists$ **a surjective** $p: V(X) \to V(Y)$**, s. t.**
if $p(x) = y$ **then** $p|_{N(x)} : N(x) \to N(y)$ **is a bijection**

X**: covering graph:** Y**: base graph;**

Vertex fibre: $p^{-1}(v)$, $v \in V(Y)$; **Edge fibre:** $p^{-1}(e)$, $e \in E(Y)$;

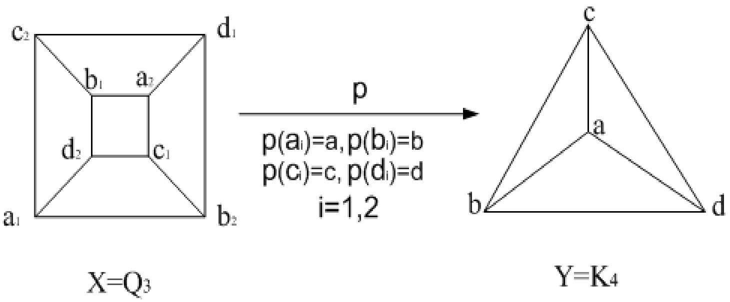
G : **the group of fibre-preserving automorphisms of** X

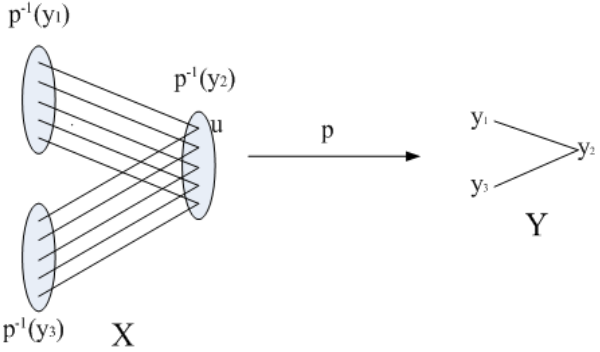
Covering transformation group K**: the kernel of** G **acting on the fibres.**

X **is connected** $\Longrightarrow$ K **acts semiregualrly on each fibre.**

Regular Cover: K **acts regularly on each fibre.**

$K \lhd G, \quad G/K \leq \mathbf{Aut}(Y)$.


c2
d1
b1
a2
d2
c1
a1
b2
X=Q3
p
p(ai)=a, p(bi)=b
p(ci)=c, p(di)=d
i=1,2
c
a
b
d
Y=K4



$p^{-1}(y_1)$
$p^{-1}(y_2)$
u
$p^{-1}(y_3)$
X
p
y_1
y_2
y_3
Y

Example

X is a bipartite graph with two biparts $V = U \cup W$,

$U = V(n, q) \setminus \{0\}$

$W = \{u + \alpha \mid \dim(\alpha) = n - 1, u \in V \setminus \alpha\}$.

where v $u + \alpha$ iff $v \in u + \alpha$.

Y is the nonincident graph of points and hyperplanes in projective geometry $\mathrm{PG}(n-1, p)$.

p: $v, \cdots kv, \cdots, (q-1)v \to \langle v \rangle$;

$u + \alpha, \cdots ku + \alpha, \cdots, (q-1)u + \alpha \to \alpha$;

Then X is a $(q-1)$-fold cover of Y.

Voltage graphs and Derived graphs

Gross and Tucker (1974).

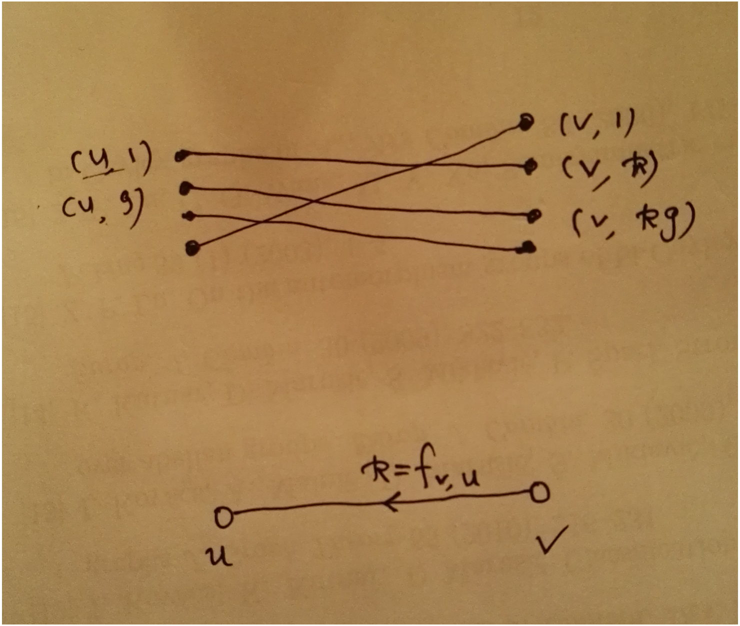
J.L. Gross and T.W. Tucker, *Topological Graph Theory,* **Wiley, New York, 1987.**

Voltage assignment f**: graph** Y**, finite group** K
a function $f : A(Y) \to K$ **s. t.** $f_{u,v} = {f_{v,u}}^{-1}$ **for each** $(u, v) \in A(Y)$.

Voltage graph: (Y, f)

Derived graph $Y \times_f K$:

vertex set $V(Y) \times K$,
arc-set $\{((u, g), (v, f_{v,u}g) \mid (u, v) \in A(Y),\ g \in K\}$.


(u, 1)
(u, g)
(v, 1)
(v, k)
(v, kg)
k=f_{v,u}
u
v

1. p**:** $(u,k) \to u$ **is a covering projection form** $Y_f \times K$ **to** Y**, whose covering transformation group is isomorphic to** K**.**

2. Each regular cover can be derived by a voltage assignment f**.**

Lifting: $\alpha \in \mathbf{Aut}(Y)$ **lifts to an automorphism** $\overline{\alpha} \in \mathbf{Aut}(X)$ **if** $\overline{\alpha} p = p\alpha$**.**

$$\begin{array}{ccc} & \overline{\alpha} & \\ X & \rightarrow & X \\ p\downarrow & & \downarrow p \\ Y & \rightarrow & Y \\ & \alpha & \end{array}$$

General Question:

Given a graph Y**, a group** K **and** $H \leq \mathbf{Aut}(Y)$**, find all the connected regular coverings** $Y \times_f K$ **on which** H **lifts.**

Note : if H **lifts to** G**, then** $G/K \cong H$**.**

A lifting problem is essentially a group extension problem

$1 \to K \to G \to H$

Lifting Theorem

Lifting Theorem: let $X = Y_f \times K$, $\alpha \in \mathbf{Aut}(Y)$. **Then**

α **lifts if and only if:** $f_W = 1 \Leftrightarrow f_{W^\alpha} = 1$, **for each closed walk** W **in** Y.

A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998), 203-218.

A. Malnic, R. Nedela, M. Skoviera, Lifting graph automorphisms by voltage assignments, European J. Combin. 21(2000), 927-947.

Abelian Covers

Theorem: Let $X = Y \times_f K$ be a connected regular cover of a graph Y, where K is abelian, If $\alpha \in$ Aut Y is an automorphism one of whose liftings $\overline{\alpha}$ centralizes K, then $f_{W^\alpha} = f_W$ for any closed W of Y.

S.F. Du, J.H.Kwak and M.Y.Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, J. Combin. Theory, B 93 (2005), 73–93.

Elementary abelian covering: $K = Z_p^n$.

S.F. Du, J.H. Kwak and M.Y. Xu, Linear criteria for lifting of automorphisms in elementary abelian regular coverings, Linear Alegebra and Its Applications, 373, 101-119(2003).

Malnic, Aleksander; Potocnik, Primoz. Invariant subspaces, duality, and covers of the Petersen graph, European J. Combin. 27 (2006), no. 6, 971(989)

Abelian covers:

1. Conder, Ma, Arc-transitive abelian regular covers of the Heawood graph. *J. Algebra* 387 (2013), 243-267.

2. Conder, Ma, Arc-transitive abelian regular covers of cubic graphs. *J. Algebra* 387 (2013), 215-242.

2. Classifying 2-arc-transitive graphs

2-arc: (u, v, w), s. t. (u, v) and (v, w) are arcs and $u \neq w$.

2–ATG X: $\mathrm{Aut}\,(X)$ acts trans. on 2-arcs of X.

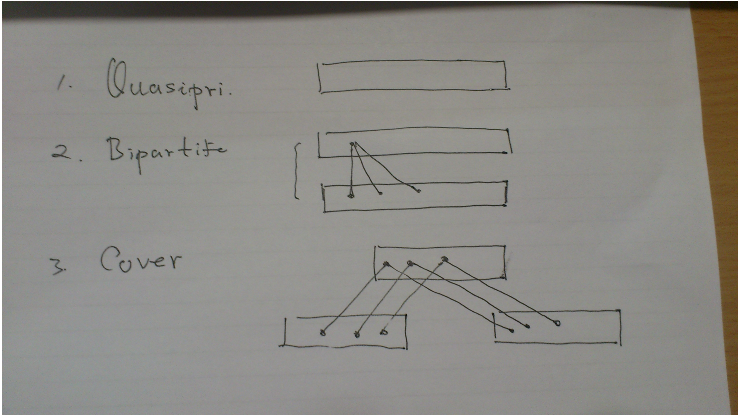
Praeger's Reduction Theorem

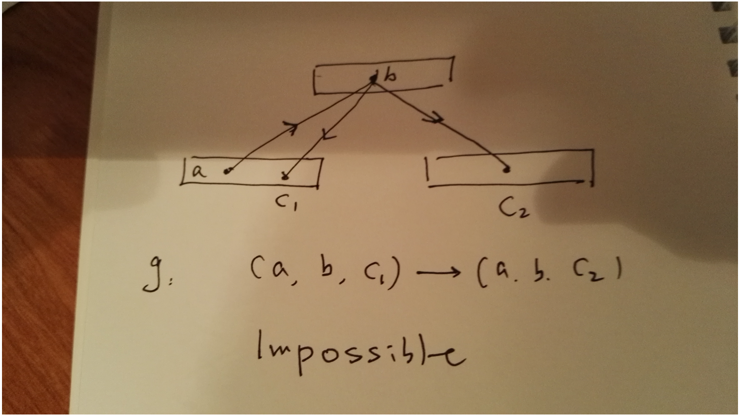
C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, *J. London Math. Soc.* (2)**47**(1993), 227-239.

Every finite connected 2-arc-transitive graphs is one of the following:

(1) *Quasiprimitive Type:* every non-trivial normal subgroup of $\mathrm{Aut}\, X$ acts transitively on $V(X)$,
Bipartite Type: every non-trivial normal subgroup of $\mathrm{Aut}\, X$ has at most two orbits on $V(X)$ and at least one of normal subgroups of $\mathrm{Aut}\, X$ has exactly two orbits on $V(X)$.

(2) *Covering Type:* There exists a normal subgroup of $\mathrm{Aut}\, X$ which has at least three orbits on $V(X) \Rightarrow$ regular covers of graphs in (1).


1. Quasipri.
2. Bipartite
3. Cover


b
a
c1
c2
g: (a, b, c1) ⟶ (a. b. c2)
Impossible

R.W.Baddeley, Two-arc transitive graphs and twisted wreath products, *J. Alg. Combin.* **2** (1993), 215–237.

A.A. Ivanov and C.E. Praeger, On finite affine 2-arc-transitive graphs, *Europ. J. Combin.* **14** (1993), 421–444.

X.G. Fang and C.E. Praeger, On graphs admitting arc-transitive actions of almost simple groups, *J. Algebra* **205** (1998), 37-52.

X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27**(1999), 3727-3754.

C. H. Li, The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$, *Trans. Amer. Math. Soc.* **353** (2001), 3511–3529.

Locally primitive graphs...and so on

Bipartite Type

A reduction theorem for this case was given by Praeger (1993).

C. E. Praeger, On a reduction theorem for finite, bipartite, 2-arc-transitive graphs, *Australas J. Combin.* **7** (1993), 21-36.

In the group theory sense, it induces two directions:

(1) Study Quasiprimitive or Bipartite type (2-ATG)

$\rightarrow$ to study finite simple group, almost simple group, Quasisimple group, primitive group, Quasiprimitive and so on, and to study the suborbit structures of the related permutation representations

$(G, G_{\alpha}, G_{\alpha,\beta})$

(2) Study regular covers of Quasiprimitive or Bipartite type (2-ATG)

$\rightarrow$ to study the group extensions of the above groups from either 'left' or 'bottom', in many cases, it is related central extension theory as well as Schur Multiplier theory, (ordinary and in most cases, modular) representations of almost simple groups and so on.

3. 2-arc-tran covers

Our long term goal is to classify the 2-arc-transitive covers by

given base graphs (2-ATG of Quasiprimitive or Bipartite type), for instance: K_n, $K_{n,n} - nK_2$, $K_{n,n}$ ...

given covering transformation groups, for instance: Z_p^n, ablian groups and metacyclic groups...

3.1 Classify 2-arc-transitive regular covers of complete graphs

Problem: Classify regular covers of complete graphs having the covering transformation group $\mathbb{Z}_p^k$ and whose fibre-preserving group acts 2-arc -transitively.

For cyclic group and $k = 2$:
S.F.Du, D.Marušič and A.O.Waller, On 2-arc-transitive covers of complete graphs, *J. Combin. Theory, B* **74** (1998), 276–290.

For $k = 3$:
S.F.Du, J.H.Kwak and M.Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, *J. Combin. Theory, B* **93** (2005), 73–93.

For $k = 4$: open

3.2 Classify 2-arc-transitive circulant and dihedrant

For a Cayley graph, its automorphism group contains a vertex-regular subgroup.

Cayley graphs of cyclic and dihedral groups are called *Circulant* and *Dihedrants*, respectively.

2-arc tran. circulant:

B.Alspach, M.D.E.Conder, D.Marušič and M.Y.Xu, A classification of 2-arc-transitive circulants, *J. Alg. Combin.*, **5** (1996), 83–86.

The proof is combinatorial and is independent on CFSG.

2-arc-tran dihedrant:

D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory, B* **87** (2003), 162–196.

S.F. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, *J. Combin. Theory, B*, 98(6), (2008), 1349-1372

3.3. 2-Arc-Transitive Metacyclic Covers of Complete Graphs

W. Q. Xu, S. F. Du, J. H. Kwak and M. Y. Xu, 2-Arc-Transitive Metacyclic Covers of Complete Graphs, *J. Combina Theory (B)*, 111 (2015), 54-74.

3.4. regular covers of $K_{n,n} - nK_1$

1. W.Q. Xu and S.F. Du, 2-Arc-trantive cyclic covers of $K_{n,n} - nK_2$, *J. Algebr. Comb.* **39**(2014), 883-902.

2. S.F. Du and W.Q. Xu, 2-arc-trantive regular covers of $K_{n,n} - nK_2$ having the covering transformation group $\mathbb{Z}_p^3$, *Journal of the Australian Mathematical Society*, 2016, 28 pages, accepted.

1. S.F. Du and W.Q. Xu, 2-Arc-trantive cyclic covers of $K_{n,n}$, submitted.

2. S.F. Du, W.Q. Xu, G.Y Yan, 2-arc-trantive regular covers of $K_{n,n}$ having the covering transformation group $\mathbb{Z}_p^2$, *Combinatorics*, accepted, 2016, 21 pages.

1-arc tran. (not necessarily 2-arc tran)

1 Pan, Huang and Liu, Arc-transitive regular cyclic covers of the complete bipartite graph $K_{p,p}$, *J. Algebraic Combin.* 42 (2015), no. 2, 619-633.

4. Relationship between topological lifting theorem and group extensions

Sabidussi Coset graph–one of basic tools for studying vertex transitive graphs:

given a group G , $H \leq G$, $a \in G$, s. t. $HaH = Ha^{-1}H$, $\langle H, a \rangle = G$.

Define a graph $Cos(G, H, HaH)$:

Vertex set $\{Hg \mid g \in G\}$, Edge set $\{H, Ha\}^G$.

Note: Every arc transitive graph can be represented by a Coset graph.

A Coset graph gives more information of groups

A voltage graph gives more clearly adjacent relations, but the properties of the groups are hidden

Except for very few cases, Lifting Theorem can be only used to determine the voltage graphs for some small base graphs

For most cases, Group Extension (group theoretical method) may be applied to determine Coset graphs.

For some cases, combining voltage graph, lifting theorem, group extension together, we may get surprising results !!!

Routine idea from group theory (coset graphs):

Try to classify the covers of Y with given group K and with a given symmetric property (*)

1. find all the some subgroups $H \leq \mathrm{Aut}(Y)$, insuring this (*)
2. determine the group extension $1 \to K \to G \to H$
3. determine coset graphs from G

Three possibilities in group theory:

1. There exists such classification for H and also it is feasible to determine the extension $1 \to K \to G \to H$

2. we do have such classification for H but it is very complicated and almost infeasible to determine the extension

3. such classification does not exist

New Idea by joining Lifting Theorem:

Instead of using the classification of H, pick up a subgroup H_1 of H, which is easy to work on ($1 \to K \to G_1 \to H_1 \to 1$), where H_1 does not need to insure ($*$)

find all Coset graphs from G_1, from which we construct voltage graphs X (sometimes a very few graphs are obtained, also their voltage assignment is very simple and nice)

for each X, choose a subgroup H_2 which insuring (*) (do not need to a classification for all such H_2), usually H_2 is bigger than H_1.

using Lifting Theorem we show H_2 lifts.

Then we find all covers.

5. Examples

By exhibiting some examples, show some applications of group theoretical tools and lifting theorem in determination of regular covers.

Example

Problem: $Y = K_5,\ V = \{0, 1, 2, 3, 4\}$

$K = (V(3, p), +)$ *find all covers s. t. the fibre preserving group acts 2-arc-tran.*

Answer:

$X(p) = K_5 \times_f K$, *either* $p = 5$ *or* $p = \pm 1 (\mod 10)$:

$f_{0,j} = (0, 0, 0)$ *for* $1 \leq j \leq 4$, $f_{1,2} = (1, 0, 0)$, $f_{1,3} = (0, 1, 0)$, $f_{2,3} = (0, 0, 1)$, $f_{1,4} = (a, b, c)$, $f_{2,4} = (-b, -c, a)$ *and* $f_{3,4} = (c, -a, -b)$, *where* $a = \frac{1+\sqrt{5}}{4}$, $b = \frac{1-\sqrt{5}}{4}$ *and* $c = \frac{\sqrt{5}}{2}$.

Proof:

1. To insure 2-arc-transitivity, A_5 should be lifted, $G/Z_p^3 = A_5$.

2. Choose a span tree, while there are 6 cotree arcs. Assign trivial voltages on tree arcs, a (3-dimensional) base to 3 cotree arcs (all possibilities), while the voltages on other 3 cotree arcs are a linely combination of the base.

3. Choose some elements of A_5, use Lifting Theorem to determine all the voltages on cotree arcs.

Note: 1. All the proof just depends on Lifting Theorem, no involving the group extensions.

2. The presentation of this voltage assignment f is not simple and nice.

Lemma

X must be isomorphic to $X(p)$.

Proof Let $X = (K_5)_f \times K$ with a voltage assignment $f : A(K_5) \to K$ and let $V(K_5) = \{0, 1, 2, 3, 4\}$. We identify K with the additive group of the 3-dimensional vector space $V(3, p)$ over $GF(p)$, where the identity element in K is identified with zero vector $\mathbf{0}$. Take a basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in K. Take the star Y_0 with the base vertex 0 as a spanning tree of K_5. Then we may assume that $f_{0,i} = \mathbf{0}$ for any $i \in V_1 := \{1, 2, 3, 4\}$.

It is easy to check that for the induced subgraph $K_5[V_1]$ of V_1 in K_5, if the voltages assigned to the respective arcs in any triangle and in any claw are linearly dependent in $K = V(3, p)$, then the group generated by all voltages has order less than p^3. This contradicts the connectedness of $(K_5)_f \times K$. So we have the following two different cases:

(1) In $K_5[V_1]$, the three voltages on the respective arcs in any triangle are linearly dependent, but there exists a claw such that the three voltages on the arcs in this claw are linearly independent. Hereafter, for any three distinct vertices i, j, k in $V(K_5)$, we use $(i,j,k)^n$ to denote the walk $(\overbrace{i,j,k,i,j,k,\cdots i,j,k}^{n\,\text{times}})$.
Without loss of generality we may assume that $f_{1,2} = \mathbf{x}$, $f_{1,3} = \mathbf{y}$, $f_{1,4} = \mathbf{z}$ and $f_{2,3} = a\mathbf{x} + b\mathbf{y}$. Take a closed walk $W = ((0,1,2)^a, (0,1,3)^b, 0, 3, 2, 0)$. We have $f_W = af_{1,2} + bf_{1,3} - f_{2,3} + (a+b)f_{0,1} + (1-a)f_{0,2} - (b-1)f_{0,3} = \mathbf{0}$. Since A_5 lifts, $f_{W^\alpha} = \mathbf{0}$ for each $\alpha \in A_5$ by Proposition **??**. Since $f_{W^{(243)}} = af_{1,4} + bf_{1,2} - f_{4,2} + (a+b)f_{0,1} + (1-a)f_{0,4} - (b-1)f_{0,2} = \mathbf{0}$, we have $f_{2,4} = -b\mathbf{x} - a\mathbf{z}$. Since $f_{W^{(12)(34)}} = \mathbf{0}$ and $f_{W^{(012)}} = \mathbf{0}$ respectively, we have

$$-a\mathbf{x} - \mathbf{z} + bf_{2,4} = \mathbf{0} \quad \text{and} \quad (a+b)\mathbf{x} + (1-b)\mathbf{y} + bf_{2,3} = \mathbf{0}. \tag{3.1}$$

Substituting the values of $f_{2,3}$ and $f_{2,4}$ in (3.1), we get the following equation system in $GF(p)$:

$$a + b^2 = 0,\ 1 + ab = 0 \quad \text{and} \quad a + b + ab = 0.$$

However, it is easy to check that this equation system has no solutions.

(2) In $K_5[V_1]$, there exists a triangle such that three voltages assigned to its arcs are linearly independent.
Without loss generality we may assume that $f_{1,2} = \mathbf{x}$, $f_{1,3} = \mathbf{y}$, $f_{2,3} = \mathbf{z}$ and $f_{1,4} = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}$. Take a closed walk $W = ((0,1,2)^a, (0,1,3)^b, (0,2,3)^c, 0, 4, 1, 0)$. Then $f_W = af_{1,2} + bf_{1,3} + cf_{2,3} - f_{1,4} + (a+b-1)f_{0,1} + (c-a)f_{0,2} - (b+c)f_{0,3} + f_{0,4} = \mathbf{0}$. Similar to (1), since $f_{W^{(132)}} = \mathbf{0}$ and $f_{W^{(123)}} = \mathbf{0}$ respectively, we have $f_{3,4} = c\mathbf{x} - a\mathbf{y} - b\mathbf{z}$ and $f_{2,4} = -b\mathbf{x} - c\mathbf{y} + a\mathbf{z}$. Since $f_{W^{(12)(34)}} = \mathbf{0}$ and $f_{W^{(02)(13)}} = \mathbf{0}$ respectively,

we have

$$-a\mathbf{x}-\mathbf{z}+cf_{1,4}+bf_{2,4}=\mathbf{0} \quad \text{and} \quad (b+c)\mathbf{x}-b\mathbf{y}+(a+b-1)\mathbf{z}+f_{2,4}-f_{3,4}=\mathbf{0}. \tag{3.2}$$

Substituting the values of $f_{1,4}$, $f_{2,4}$ and $f_{3,4}$ in (3.2), we get the following equation system in $GF(p)$:

$$-a+ac-b^2=0, \quad -1+c^2+ab=0, \quad a-b-c=0 \quad \text{and} \quad 2a+2b-1=0$$

Solving this equation system, we get $4a^2-2a-1=0, \quad b=\frac{1}{2}-a \quad \text{and} \quad c=2a-\frac{1}{2}$. However, the first equation has a solution if and only if $p=5$ or $p \equiv \pm 1 \pmod{10}$. If $p \equiv \pm 1 \pmod{10}$, we have two solutions; $(a,b,c)=(\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{\sqrt{5}}{2})$ and $(a,b,c)=(\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}, \frac{-\sqrt{5}}{2})$. The graph determined by the first solution for (a,b,c) is precisely $X_2(p)$. It is easy to see that the graph determined by the second solution for (a,b,c) is isomorphic to $X_2(p)$ if we assume $f_{1,2}=\mathbf{y}$ and $f_{1,3}=\mathbf{x}$. If $p=5$, we have $(a,b,c)=(-1,-1,0)$. $\square$

Lemma

For a graph $X(p)$, the fibre-preserving group of automorphisms acts 2-arc-transitively on the graph.

Proof Since $\overline{A}$ is isomorphic to either A_5 or S_5, it suffices to show that A_5 lifts. Since A_5 is generated by (13)(24) and (012), A_5 lifts if and only if these two generators lift.
Let W be a closed walk in K_5 with $f_W = \mathbf{0}$. We may assume that the arc (i,j) (resp. (j,i)) appears $\ell_{i,j}$ (resp. $\ell_{j,i}$) times in W and let $t_{i,j} = \ell_{i,j} - \ell_{j,i}$. Since $f_{i,j} = -f_{j,i}$, we get $t_{i,j} = -t_{j,i}$. Then $f_W = \sum_{0 \leq i < j \leq 4} t_{i,j} f_{i,j} = \mathbf{0}$.

Substituting the values of $f_{i,j}$ in it, we get the following three relations between $\{t_{i,j}\}$;

$$\begin{aligned} t_{1,2} &= -at_{1,4} + bt_{2,4} - ct_{3,4}, \\ t_{1,3} &= -bt_{1,4} + ct_{2,4} + at_{3,4}, \\ t_{2,3} &= -ct_{1,4} - at_{2,4} + bt_{3,4}. \end{aligned} \tag{3.3}$$

Since W is a closed walk, the numbers of arcs in W coming from i and going into i are equal for any vertex i in $V(K_5)$. So we get

$$\begin{aligned} t_{0,1} &= t_{1,2} + t_{1,3} + t_{1,4} = (1-a-b)t_{1,4} + (b+c)t_{2,4} + (a-c)t_{3,4}, \\ t_{0,2} &= t_{2,1} + t_{2,3} + t_{2,4} = (a-c)t_{1,4} + (1-a-b)t_{2,4} + (c+b)t_{3,4}, \\ t_{0,3} &= t_{3,1} + t_{3,2} + t_{3,4} = (b+c)t_{1,4} + (a-c)t_{2,4} + (1-a-b)t_{3,4}, \\ t_{0,4} &= t_{4,1} + t_{4,2} + t_{4,3} = -t_{1,4} - t_{2,4} - t_{3,4}. \end{aligned} \tag{3.4}$$

Let $\alpha = (13)(24)$. Then

$$f_{W^\alpha} = \sum_{0 \leq i < j \leq 4} t_{i,j} f_{i^\alpha, j^\alpha}$$

$= \mathrm{t}_{1,2} f_{3,4} + t_{1,3} f_{3,1} + t_{2,3} f_{4,1} + t_{1,4} f_{3,2} + t_{2,4} f_{4,2} + t_{3,4} f_{1,2}$.
Substituting the values of $f_{i,j}$ in it and by using (3.3), we get

$$\begin{aligned} f_{W^\alpha} &= (ct_{1,2} - at_{2,3} + bt_{2,4} + t_{3,4})\mathbf{x} \\ &\quad +(-at_{1,2} - t_{1,3} - bt_{2,3} + ct_{2,4})\mathbf{y} \\ &\quad +(-bt_{1,2} - ct_{2,3} - t_{1,4} - at_{2,4})\mathbf{z} \\ &= \left((bc + a^2 + b)t_{2,4} - (c^2 + ab - 1)t_{3,4}\right)\mathbf{x} \\ &\quad + \left((a^2 + b + bc)t_{1,4} + (ac - a - b^2)t_{3,4}\right)\mathbf{y} \\ &\quad + \left(-(1 - ab - c^2)t_{1,4} + (-b^2 + ac - a)t_{2,4}\right)\mathbf{z}. \end{aligned} \tag{3.5}$$

Since $(a, b, c) = (\frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4}, \frac{\sqrt{5}}{2})$, it is easy to check that $bc + a^2 + b = c^2 + ab - 1 = ac - b - b^2 = 0$. Hence $f_{W^\alpha} = \mathbf{0}$ and so α lifts by Lifting Theorem.

Let $\beta = (012)$. Similarly, we may prove that $f_{W^\beta} = \mathbf{0}$. (Here (3.3) and (3.4) are used again and the details are omitted.) Thus we prove that β also lifts by Lifting Theorem. Since the two generators α and β of A_5 lift, A_5 also lifts. $\square$

Essentially, it is group extension problem: $G/\mathbb{Z}_p^3 = A_5$.

Where is the group theory and representation theory ?
Hidden !!

There must be a deep relation between group theory and topological method (Lifting Theorem), under the combinatorial frame.

Example 5.2: Construct voltage graphs from abstract groups

Main idea:

We may determine the group extension $1 \to K \to G \to H \to 1$, depending on a lots of group theoretical tools, including central extension theory and modular representations of groups.

Study the permutation representations of G

Find the coset graphs

Find the voltage graphs from the known coset graphs, with simple and nice f

Use Lifting Theorem to show a subgroup (insuring your properties, as bigger as possible) of $\mathrm{Aut}\,(Y)$ lifts.

Example

Let $Y = K_{1+p}$ where $V(Y) = PG(1,p) = GF(p) \cup \{\infty\}$ and let $K = (V(3,p),+)$. Find all the regular coverings $X = Y \times_f K$ such that $\mathrm{PGL}(2,p) \leq \mathrm{Aut}\,(Y) = S_{p+1}$ lifts.

Solution: (1) Define $X(p) =: K_{1+p} \times_f Z_p^3$ as follows:
$f_{\infty,j} = (0,1,2j)$,
$f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $GF(p)$.

(2) $X'(5) = K_6 \times_f Z_p^3$ as follows:

....

Proof of Example 5.2

$A/K \cong \mathrm{PGL}(2,p)$ for $p \geq 5$ and $n = 1 + p$.
Take a fibre F and a vertex $v \in F$. Then $A_F = A_v K$.
Since $(|A : A_F|, |K|) = (1 + p, p^3) = 1$ and K is an abelian normal subgroup of A, we know that K has a complement in A which is isomorphic to $\mathrm{PGL}(2,p)$, that is

$$A \cong Z_p^3 \rtimes \mathrm{PGL}(2,p)$$

Step 1: Determination of structure of the group A

Modular p- Representations of 2-dimensional linear groups:

1. Brauer and C. Nesbitt, On the modular characters of groups, Annals of Math, 42(2), 556-590.

2.R. Burkhardt, Die Zerlegungsmartrizen de Gruppen $PSL(2, p^f)$, J. Algebra of Algebra, 40(1976), 75-96

$\mathrm{SL}(2, p)$ has p irreducible modular p- Representations

$\mathrm{PSL}(2, p)$ has $\frac{p+1}{2}$ irreducible modular p- Representations with degrees $1, 3, 5, \cdots, p$

Degree 3:
$V_3 = \langle x^i y^j \mid i+j=2 \rangle$ homogeneous space over F_p
$g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
Define $G = \mathrm{PSL}(2,p)$-module V_3 extended by
$g(x^i y^j) = (a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$

Let $G = PGL(2,p)$. Define two G-modules V_3 extended by
$g(x^i y^j) = det(g)^{-1}(a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$
and
$g(x^i y^j) = det(g)^{\frac{p-1}{2}-1}(a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$

Take a base in V_3, we get two homomorphisms ϕ of $\mathrm{PGL}(2, p)$ into $\mathrm{GL}(3, p)$

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad - bc)^{-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad + bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad - bc)^{\frac{p-1}{2}-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad + bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$

Note: The first case will give the covers $X(p)$
the second will gives the covers $X'(5)$.

Step 2: Determination of conjugacy class of point stabilizers

Take a subgroup $H_1 = \langle t_1 \rangle \rtimes \langle a_1 \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ of $\mathrm{PGL}(2, p)$, where

$$t_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

for a generator θ of $GF(p)^*$. Let $PG(1, p) = \{\infty, 0, 1, \ldots, p-1\}$ be the projective line over $GF(p)$, where we identify $\langle(0, 1)\rangle$ and $\langle(1, \ell)\rangle$ with ∞ and ℓ, respectively. Then, H_1 fixes $\infty \in PG(1, p)$ and t_1^i maps ℓ into $\ell + i$. Furthermore, we have $H := \phi(H_1) = \langle t \rangle \rtimes \langle a \rangle$, where $t = \phi(t_1)$ and $a = \phi(a_1)$, and for any i,

$$t^i = \phi(t_1^i) = \begin{pmatrix} 1 & 2i & 2i^2 \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a^i = \phi(a_1^i) = \begin{pmatrix} \theta^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^{-i} \end{pmatrix}.$$

Lemma

Let $M = K \rtimes H$. Then, M has only one conjugate class of subgroups L satisfying $\langle a \rangle \leq L \cong H$ and $L \cap K = 1$.

Proof Note that $|M| = |K \rtimes H| = |(K \rtimes \langle t \rangle) \rtimes \langle a \rangle| = p^4(p-1)$. Let $P = K \rtimes \langle t \rangle$. Then, P is a p-group of order p^4. Since $p \geq 5$ by assumption, P is a regular p-group (for the definition of regular p-groups. Since $\Phi(P) \leq K$ and the order of t is p, P has exponent p. Clearly, M has only one conjugacy class of subgroups isomorphic to $\langle a \rangle$. Assume that L is a subgroup of M such that $\langle a \rangle \leq L \cong H$ and $L \cap K = 1$. Then, we may assume that $L = \langle kt \rangle \rtimes \langle a \rangle$ for some $k = (x, y, z) \in K$. Suppose that $(kt)^a = (kt)^i$. Then, we have $(kt)^a = k^a t^a = (\theta x, y, \theta^{-1} z) t^{\theta^{-1}}$

$$\begin{aligned}(kt)^i &= (kk^{t^{-1}}k^{t^{-2}}\cdots k^{t^{-i+1}})t^i\\ &= ((x,y,z)+(x,-2x+y,2x-2y+z)+\cdots\\ &\quad +(x,-2(i-1)x+y,2(i-1)^2x-2(i-1)y+z))t^i\\ &= (ix,-(i-1)ix+iy,\tfrac{(i-1)i(2i-1)}{3}x-(i-1)iy+iz)t^i.\end{aligned}$$

Thus, we get $i=\theta^{-1}$ and

$$(\theta x, y, \theta^{-1}z) = (ix, -(i-1)ix+iy, \frac{(i-1)i(2i-1)}{3}x-(i-1)y+iz).$$

From these two equations, we have $\theta x = ix = \theta^{-1}x$ and so $\theta^2 x = x$. Since $p \geq 5$, we get $\theta^2 \neq 1$, and so $x=0$ and $y=0$ by the second equation again. Hence, $k=(0,0,z)$ for any $z \in GF(p)$, that means k has p possibilities. For each k, we get an $L = \langle kt \rangle \rtimes \langle a \rangle$; in particular, $L=H$ when $z=0$. Furthermore, these p subgroups are conjugate in M.

In fact, for any $k = (0, 0, z)$, by taking $k' = (0, \frac{z}{2}, 0)$, we have

$$\begin{aligned} (kt)^{k'} &= k(k')^{-1}tk' = k(k')^{-1}(k')^{t^{-1}}t \\ &= ((0,0,z) - (0, \tfrac{z}{2}, 0) + (0, \tfrac{z}{2}, -z))t = (0,0,0)t = t \end{aligned}$$

$$a^{k'} = {k'}^{-1}ak' = {k'}^{-1}(k')^{a^{-1}}a = \left((0, -\frac{z}{2}, 0) + (0, \frac{z}{2}0)\right) a = a,$$

which forces $L^{k'} = H$, completing the proof. □

Lemma

Let $[A : H]$ be the set of right cosets of H in A. Then, in its right multiplication action on $[A : H]$, A has $p-1$ suborbits of length p not contained in $[M : H]$, which correspond to the $p-1$ double cosets $Hg(0, y, 0)H$ for any $y \in GF(p)^*$ and $g = \phi(g_1)$, where

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof Suppose that the double coset D corresponds to a suborbit of A of length p relative to H not contained in $[M : H]$. Since H has only one conjugacy class of subgroups of order $p-1$, a must fix a point in this suborbit.
Noting that T is 2-transitive on $[T : H]$, we may choose $D = HgkH$ such that $Hgk = Hgka$, in other words, $Hg = Hga^{-1}k^a$, which forces that $Hg = Hga^{-1}$ and $k^a = k$. Hence, we may fix $g = \phi(g_1)$. Assume $k = (x, y, z)$. From $(\theta x, y, \theta^{-1}z) = k^a = k = (x, y, z)$, we have $x = z = 0$ as $\theta \neq \pm 1$, and so $k = (0, y, 0)$, where $y \neq 0$. Therefore, we get $p-1$ choices for k and so for D. □

Step 4: Determination of Coset graphs

Now, $M = K \rtimes H = A_F$ for a fibre F. For any $u \in F$, we have $M_u \cong H$ and $M_u \cap K = 1$. Since M has only one conjugacy class of subgroups isomorphic to $\langle a \rangle$, there exists a vertex $v \in F$ such that $\langle a \rangle \leq M_v$. By Lemma **??**, M_v is conjugate to H in M. It follows that H fixes a vertex in F. Therefore, X is isomorphic to one of $X(A, H, D)$, where $D = Hg(0, y, 0)H$ is as in Lemma 0.6. Moreover, it is easy to see that the $p-1$ graphs corresponding to the $p-1$ choices for D are isomorphic to each other, by changing the basis of $V(3,p)$. Now, we may choose $k = (0,1,0)$. Note that

$$g = \phi(g_1) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Since $(gk)^2 = 1$, we get $D = HgkH = H(gk)^{-1}H = D^{-1}$. So, $X(A, H, D)$ is an undirected graph. Clearly, A acts 2-arc-transitively on $X(A, H, D)$, because T is 3-transitive on $V(K_n)$.

Lemma

$X(A, H, D) \cong X(p)$, and its group of fibre-preserving automorphisms acts 2-arc-transitively.

Proof Considering the action of $\mathrm{PGL}(2, p)$ on $PG(1, p)$, one can easily check that for $\ell \in GF(p)^*$ both $g_1 t_1^{\ell} g_1 t_1^{i}$ and $g_1 t_1^{i-\ell^{-1}}$ map ∞ to $i - \ell^{-1}$, respectively. Since $(\mathrm{PGL}(2, p))_\infty = H_1$, we have that for any $i \in GF(p)$, $H_1 g_1 t_1^{\ell} g_1 t_1^{i} = H_1 g_1 t^{i-\ell^{-1}}$ and so under the homomorphism ϕ mentioned before $Hgt^{\ell}gt^{i} = Hgt^{i-\ell^{-1}}$. In addition, $(Hg)gt^{i} = H$.

By the arguments before the lemma, we know that in the coset graph $X(A, H, D)$, H is adjacent to $Hgkt^{\ell}$ for any $\ell \in GF(p)$.

Hence, for any $i \in GF(p)$, Hgt^{i} is adjacent to $Hgkt^{\ell}gt^{i} = Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})}$ for any $\ell \in GF(p)$.

If $\ell = 0$, then

$$Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} = H(0,1,0)gt^{i} = H(0,-1,-2i).$$

Hence, Hgt^{i} is adjacent to $H(0,-1,-2i)$ for any $i \in GF(p)$, or equivalently, H is adjacent to $Hgt^{j}(0,1,2j)$ for any $j \in GF(p)$. Assume $\ell \in GF(p)^{*}$ and let $i - \ell^{-1} = j$. Then,

$$\begin{aligned} Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} &= Hgt^{i-\ell^{-1}}(0,1,0)t^{\ell}gt^{i} \\ &= Hgt^{i-\ell^{-1}}(\ell, 2i\ell - 1, 2i^{2}\ell - 2i) = Hgt^{j}\left(\tfrac{1}{i-j}, \tfrac{i+j}{i-j}, \tfrac{2ij}{i-j}\right.\end{aligned}$$

Hence, Hgt^{i} is adjacent to $Hgt^{j}\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for any $i \neq j \in GF(p)$.

Considering the action of $\mathrm{PGL}(2,p)$ on $PG(1,p)$, we may define a bijection from $[\mathrm{PGL}(2,p) : H_1]$ to $PG(1,p)$ by sending H_1 to ∞ and $H_1 g_1 t^{i}$ to i. Accordingly, we may define a bijection from $[T : H]$ to $PG(1,p)$ by sending H to ∞ and Hgt^{i} to i.

Finally, we may define a map σ from $V(X(A, H, D))$ to $V(X(p)) = PG(1, p) \times K$ by sending Hk to (∞, k) and $Hgt^i k$ to (i, k). In viewing the above arguments and the definition of $X(p)$, we find that σ is an isomorphism from $X(A, H, D)$ to $X(p)$. Moreover, since A acts 2-arc-transitively on $X(A, H, D)$, it follows that for the graph $X(p)$, its group of fibre-preserving automorphisms acts 2-arc-transitively. □

Step 6: Generalize to $X(p)$ to $X(q)$, using Lifting Theorem

Lemma

For each cover in $X(q)$, the group of fibre-preserving automorphisms acts 2-arc-transitively.

Proof Recall that $V(K_{1+q})$ is identified with the projective line $PG(1,q) = GF(q) \cup \{\infty\}$. We will adopt the usual computations between ∞ and the elements in $GF(q)$, that is, $\infty + i = \infty$ for $i \in GF(q)$; $\infty i = \infty$ for $i \in GF(q)^*$; and $\frac{\infty}{\infty} = 1$. Let K be the corresponding additive group of $V(3,q)$. Then, $X(q) = K_{1+q} \times_f K$ is defined by $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $PG(1,q)$.
To prove the lemma, it suffices to show that $\mathrm{PGL}(2,q)$ lifts. For a computation, we identify the element ∞ and any $i \in GF(q)$ in $PG(1,q)$ with $\langle(1,0)\rangle$ and $\langle(i,1)\rangle$ respectively. For a matrix g in $\mathrm{GL}(2,q)$, we denote by $\overline{g}$ the image of g in $\mathrm{PGL}(2,p^\ell)$ under the natural homomorphism.

Then, the action of $\overline{g} \in \mathrm{PGL}(2, p^\ell)$ on ∞ and any $i \in PG(1, p^\ell)$ can be written respectively as follows:

$$\infty^{\overline{g}} := \langle g(1,0) \rangle \quad \text{and} \quad i^{\overline{g}} := \langle g(i,1) \rangle.$$

Let

$$g_1 = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \ g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ g_4 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

where x is a primitive element in $GF(q)$. Then, all of these elements generate $\mathrm{PGL}(2, q)$. In addition, it is easy to check that

$$i^{\overline{g_1}} = ix^2, \quad i^{\overline{g_2}} = i + 1, \quad i^{\overline{g_3}} = \frac{i}{i+1}, \quad i^{\overline{g_4}} = ix,$$

where $i \in PG(1, q)$. In what follows, we show that for $1 \leq k \leq 4$, $\overline{g_k}$ lifts.

Let W be a closed walk in Y with $f_W = 0$, and for any arc $(i,j) \in A(Y)$, let $\ell_{i,j}$ has the same notation as above.

Now, we get

$$f_W = \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i,j} = \sum_{(i,j)\in A(Y)} \ell_{i,j}\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right) = \mathbf{0}.$$

Therefore, we have

$$\sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} = 0.$$

Also, we have

Now, we get

$$
\begin{aligned}
f_{W^{\overline{g_1}}} &= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i^{\overline{g_1}}, j^{\overline{g_1}}} \\
&= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{ix^2, ix^2} \\
&= \sum_{(i,j)\in A(Y)} \ell_{i,j} \left(\frac{1}{ix^2 - jx^2}, \ \frac{ix^2 + jx^2}{ix^2 - jx^2}, \ \frac{2ix^2jx^2}{ix^2 - jx^2} \right) \\
&= \left(x^{-2} \sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j}, \ \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j}, \ x^2 \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} \right) \\
&= \mathbf{0}.
\end{aligned}
$$

Similarly, we get that $f_{W^{\overline{g_k}}} = \mathbf{0}$, for $k = 2, 3$ and 4. Then $\overline{g_k}$ lifts, and so $\mathrm{PGL}(2, q)$ lifts. □

Main Idea:

When meeting some difficulties from usual group theoretical analysis, we try to start from a small subgroup of $\mathrm{Aut}(Y)$, to find nice voltage assignments f, so that Lifting Theorem can be possibly and easily used to show our desired groups (insuring our symmetrical properties) lift.

Question: $Y = K_{n,n}$, $K = \mathbb{Z}_p^2$, find the covers $X = Y_f \times K$ such that the fibre-preserving subgroup acts 2-arc-transitively

$V(Y) = U \cup V$

$\mathrm{Aut}\,(Y) = (S_n \times S_n) \rtimes Z_2$.

A : a 2-arc-transitive subgroup, $G \leq A$: fixing two biparts

$\widetilde{A}/K \cong A$, $\widetilde{G}/K \cong G$

G_u acts 2-tran. on $W \Longrightarrow G$ is a 2-transitive group of X on W and so on U.

Here consider a special case:

G^U is an affine group: $G^U \leq \mathrm{AGL}(s,p) = \mathbb{Z}_p^s \rtimes \mathrm{GL}(s,p)$

$Y = K_{p^s,p^s}$, $s \geq 2$, $K = \mathbb{Z}_p^2$, $V = U \cup W$

$$U = \{\alpha \mid \alpha \in V(s,p)\},\ W = \{\alpha' \mid \alpha \in V(s,p)\},$$

$T_U \cong T_W \cong \mathbb{Z}_p^s$:

$\widetilde{T}_U/K = T_U$, $\widetilde{T}_W/K = T_W$, $\widetilde{T}/K = (T_U \times T_W)/K$.

$G = (T_U \times T_W) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$

$A = G\langle\sigma\rangle$, σ exchanges two biparts of $K_{n,n}$.

Group problem:

$\widetilde{G}/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$,

where H is tran on $V(s,p) \setminus \{0\}$.

Usual Way:

1. Determine p-subgroups P of $\widetilde{G}$ such that $P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$;
2. Determine $\widetilde{G} = P.\widetilde{H}$, where $\widetilde{H}/K = H$.

$P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$.

$c = 2$, $\exp(P) = p$, $Z(P) = P' = \mathbb{Z}_p^2$.

About meta-abelian p-groups,

1. $P' = \mathbb{Z}_p$, extra-special p-group

2. $P' = \mathbb{Z}_p^k$,
Sergeicuk, V. V. The classification of metabelian p -groups. (Russian) Matrix problems (Russian), pp. 150-161. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977.

Visneveckii, A. L., Groups class 2 and exponent p with commutator group $\mathbb{Z}_p^2$, Doll, Akad. Nauk Ukrain. SSR Ser, 1980, No 9, 9-11. 1980.

Scharlau, Rudolf, Paare alternierender Formen. Math. Z. 147 (1976), no. 1, 13-19.

$\widetilde{G}/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H,\ H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p),\ \widetilde{G} = P.\widetilde{H}$

Transitive subgroups H_1 of $\mathrm{GL}(s,p)$:

$\mathrm{SL}(d,q) \leq H_1 \leq P\Gamma L(d,q),\ q^d = p^s$
$Sp(d,q) \lhd H_1,\ q^{2d}$
$G_2(q) \lhd H_1,\ q^6$
$SL(2,3) \lhd H_1,\ q = 5^2, 7^2, 11^2, 23^2$
$A_6,\ 2^4$
$A_7,\ 2^4$
$PSU(3,3),\ 2^6$
$SL(2,13),\ 3^3$.

1. Huppert, Bertram Zweifach transitive, auflsbare Permutationsgruppen. (German) Math. Z. 68 1957 126-150.

2. Hering, Christoph, Transitive linear groups and linear groups which contain irreducible subgroups of prime order. Geometriae Dedicata 2 (1974), 425-460.

3. Hering, Christoph Zweifach transitive Permutationsgruppen, in denen 2 die maximale Anzahl von Fixpunkten von Involutionen ist. (German) Math. Z. 104 1968 150-174.

$$\widetilde{T}/K = \mathbb{Z}_p^s \times \mathbb{Z}_p^s,$$

$$\widetilde{T} = \langle \widetilde{T}_{\widetilde{w}}, \widetilde{T}_{\widetilde{u}} \rangle = (K \times \widetilde{T}_{\widetilde{w}}) \rtimes \widetilde{T}_{\widetilde{u}},$$

$$K = \langle z_1, z_2 \rangle = \widetilde{T}' = Z(\widetilde{T}) \cong \mathbb{Z}_p^2,$$

$$L := \widetilde{T}_{\widetilde{w}} = \langle a_i \mid 1 \leq i \leq s \rangle, \quad R := \widetilde{T}_{\widetilde{u}} = \langle b_i \mid 1 \leq j \leq s \rangle,$$

$$[a_i, b_j] = z_1^{\alpha_{ij}} z_2^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{F}_p,$$

$$\mathrm{A} := (\alpha_{ij})_{s\times s} \quad \text{and} \quad \mathrm{B} := (\beta_{ij})_{s\times s}.$$

For any $\ell = \Pi_{i=1}^s a_i^{\alpha_i} \in L$ and $r = \Pi_{i=1}^s b_i^{\beta_i} \in R$,

$$[\ell, r] = z_1^{\alpha \mathrm{A} \beta^{\mathrm{T}}} z_2^{\alpha \mathrm{B} \beta^{\mathrm{T}}},$$

Theorem

We may take $\mathrm{A} = \mathrm{I}$ *and* $\mathrm{B} = \mathrm{M}$,

$$\mathrm{M_d} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{pmatrix}_{d \times d}$$

$$\mathrm{M} = \begin{pmatrix} \mathrm{M_d} & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M_d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M_d} \end{pmatrix}_{s \times s},$$

where $d \geq 2$, $d \mid s$ *and* $\varphi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ *is an irreducible polynomial of degree* d *over* $\mathbb{F}_p$.

$$X \cong X(s, p, \varphi(x)) = Y \times_f K : f_{\alpha,\beta'} = (\beta\alpha^T, \beta \mathrm{M} \alpha^T),$$

Proof

Step 1: Show that $|\mathrm{A}|, |\mathrm{B}| \neq 0$.

Consider the quotient graph induced by $\langle z_1^i z_2^j \rangle$ of order p, which is a p-fold cover of K_{p^s,p^s}.

Then $\widetilde{T}/\langle z_1^i z_2^j \rangle$ is an extraspecial p-group and $Z(\widetilde{T}/\langle z_1^i z_2^j \rangle)$ is of order p.

Take $i = 1$ and $j = 0$. In $\widetilde{T}/\langle z_1 \rangle$, we have $[\bar{\ell}, \bar{r}] = \bar{z}_2{}^{\alpha \mathrm{B} \beta^{\mathrm{T}}}$

If $|\mathrm{B}| = 0$, them take $\beta_1 \neq 0$ such that $\mathrm{B}\beta_1^{\mathrm{T}} = 0$, which implies $\alpha \mathrm{B} \beta_1^{\mathrm{T}} = 0$ for any α. Therefore, for the corresponding element r_1, we have $[\bar{\ell}, \bar{r}_1] = \bar{1}$ for any ℓ.

Now, $\bar{r}_1 \in Z(\widetilde{T}/\langle z_1 \rangle) \setminus (K/\langle z_1 \rangle)$ and so $Z(\widetilde{T}/\langle z_1 \rangle)$ is of order at least p^2, a contradiction.

Hence, $|\mathrm{B}| \neq 0$. Similarly, $|\mathrm{A}| \neq 0$.

Step 2: Show that $\mathrm{A} = \mathrm{I}$.

For $\mathrm{P} = (p_{ij})_{s\times s}$, $\mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s,p)$, set
$a_i' = \prod_{\ell=1}^{s} a_\ell^{p_{\ell i}} \quad \text{and} \quad b_j' = \prod_{\ell=1}^{s} b_\ell^{q_{\ell j}}$.
$[a_i', b_j'] = z_1^{\alpha_{ij}'} z_2^{\beta_{ij}'}$,

where $(\alpha_{ij}')_{s\times s} = \mathrm{P}^{\mathrm{T}}\mathrm{A}\mathrm{Q}, \quad (\beta_{ij}')_{s\times s} = \mathrm{P}^{\mathrm{T}}\mathrm{B}\mathrm{Q}$.
Take $\mathrm{P} = (\mathrm{A}^{-1})^{\mathrm{T}}$ and $\mathrm{Q} = \mathrm{I}$. Then we get $(\alpha_{ij}')_{s\times s} = \mathrm{I}$.
Hence, assume

$[\ell, r] = z_1^{\alpha\beta^{\mathrm{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathrm{T}}}$.

Step 3: Find the conditions for the matrix B.

Recall H lifts to $\widetilde{H}$ and $\widetilde{G} = ((K \times L) \rtimes R)\widetilde{H}$. Then for any $\widetilde{h} \in \widetilde{H}$, set

$$a_i^{\widetilde{h}} = (\Pi_{j=1}^{s} a_j^{p_{ji}})k_{i1}, \quad b_i^{\widetilde{h}} = (\Pi_{j=1}^{s} b_j^{q_{ji}})k_{i2}, \quad z_1^{\widetilde{h}} = z_1^a z_2^b, \quad z_2^{\widetilde{h}} = z_1^c z_2^d, \tag{1}$$

where $i = 1, 2, \cdots s$, $k_{i1}, k_{i2} \in K$ and moreover, set

$$\mathrm{P} = (p_{ij})_{s\times s}, \quad \mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s, p).$$

Since $[\ell, r] = z_1^{\alpha\beta^{\mathrm{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathrm{T}}}$, we have

$$[\ell^{\widetilde{h}}, r^{\widetilde{h}}] = z_1^{\alpha\mathrm{P}^{\mathrm{T}}\mathrm{Q}\beta^{\mathrm{T}}} z_2^{\alpha\mathrm{P}^{\mathrm{T}}\mathrm{B}\mathrm{Q}\beta^{\mathrm{T}}} = z_1^{a\alpha\beta^{\mathrm{T}}+c\alpha\mathrm{B}\beta^{\mathrm{T}}} z_2^{b\alpha\beta^{\mathrm{T}}+d\alpha\mathrm{B}\beta^{\mathrm{T}}},$$

which forces that

$$\mathrm{P}^{\mathrm{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B}, \quad \mathrm{P}^{\mathrm{T}}\mathrm{B}\mathrm{Q} = b\mathrm{I} + d\mathrm{B}.$$

Then we have

$$(a\mathrm{I} + c\mathrm{B})\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (b\mathrm{I} + d\mathrm{B}). \tag{2}$$

$\varepsilon : \widetilde{h} \to \mathrm{Q}$ gives an homomorphism from $\widetilde{H}$ to $\mathcal{H} := \varepsilon(\widetilde{H})$. Then $\mathcal{H}$ acts transitively on $V \backslash \{0\}$.

Let $L = \{f(\mathrm{B}) \mid f(x) \in \mathbb{F}_p[x]\}$, a subalgebra of $\mathrm{Hom}_{\mathbb{F}_p}(V, V)$.

Let $L^* = \{f(\mathrm{B}) \in L \mid |f(\mathrm{B})| \neq 0\} \subset L$.

Then L^* forms a group of $\mathrm{GL}(s, p)$ (finiteness of L) .

Since $\mathrm{P}^{\mathrm{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B} \in L^*$, we have $(a\mathrm{I} + c\mathrm{B})^{-1}$ is contained in L^*.

$$\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (a\mathrm{I} + c\mathrm{B})^{-1}(b\mathrm{I} + d\mathrm{B}) \in L^*.$$

That is, $\mathcal{H}$ normalizes L.

Step 4: Show L is a field.

Consider L-right module V. For any $v \in V$, vL is irreducible.

In fact, let V_1 be an irreducible L-submodule of vL.
Take $g \in \mathcal{H}$ such that $vg \in V_1$. Then
$\dim(V_1) \leq \dim(vL) = \dim(vLg) = \dim(vgL) \leq \dim(V_1 L) = \dim(V_1)$.
Hence, $\dim(V_1) = \dim(vL)$, that is $vL = V_1$.

Take any $\ell \in L \setminus L^*$. Then $v\ell = 0$ for some $v \in V \setminus \{0\}$ and so $(vL)\ell = v\ell L = 0$. For any $w \in V \setminus vL$, we have $vL \neq wL$ If $wL = (v+w)L$, then $v \in wL$ forcing $wL = vL$, a contradiction. Therefore, $wL \neq (v+w)L$, which means $wL \cap (v+w)L = \{0\}$. Since $v\ell = 0$, we have
$w\ell = v\ell + w\ell = (v+w)\ell \in wL \cap (v+w)L = \{0\}$. By the arbitrary of $w \in V \setminus vL$ and $(vL)\ell = 0$, we get $u\ell = 0$ for any vector $u \in V$ and so $\ell = 0$.

Therefore, L is a field

Step 5: Determination of B.

Let $p(x) = \sum_{i=0}^{d} a_i x^i$ be the minimal monic polynomial for B. Since $L = \mathbb{F}_p(B)$ is a field, $p(x)$ is irreducible, and $\mathrm{I}, \mathrm{B}, \mathrm{B}^2, \cdots \mathrm{B}^{d-1}$ is a base of L over $\mathbb{F}_p$.

Set $V = \bigoplus_i v_i L$, where every $v_i L$ is an irreducible L-module of dimension d. Clearly, $d \Big| s$ so that $1 \le i \le \frac{s}{d}$.

Define $\mathcal{B}(v) = v\mathrm{B}$ for any $v \in V$. Then $(e_1, \cdots, e_s)\mathcal{B} = (e_1, \cdots e_s)\mathrm{B}^{\mathrm{T}}$, $e_1, \cdots e_s$ are unit vectors.

V has a base:

$$v_1, v_1\mathrm{B}, \cdots, v_1\mathrm{B}^{d-1}; v_2, v_2\mathrm{B}, \cdots, v_2\mathrm{B}^{d-1}; \cdots; v_{\frac{s}{d}}, v_{\frac{s}{d}}\mathrm{B}, \cdots, v_{\frac{s}{d}}\mathrm{B}^{d-1}.$$

Under this base, the matrix of $\mathcal{B}$ is exactly M. Therefore, $\mathrm{B} \sim \mathrm{B}^{\mathrm{T}} \sim \mathrm{M}$, and we may let $\mathrm{B} = \mathrm{M}$.

Step 6: Show X is isomorphic to $X(s,p,\varphi(x)) = Y \times_f K$, $f_{\alpha,\beta'} = (\beta\alpha^{\mathrm{T}}, \beta\mathrm{M}\alpha^{\mathrm{T}})$.

$X \cong X_1 := \mathbf{B}(\widetilde{T}, L, R; RL)$, recall $\widetilde{T} = (K \times L) \rtimes R$.

Connectedness and valency: $\langle (LR)(LR)^{-1} \rangle = \langle L, R \rangle = \widetilde{T}$ and $|RL : L| = p^s$

Cover: the quotient graph $\overline{X}_1$ induced by the center K is K_{p^s,p^s}.

For any $\ell = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_s^{\alpha_s} \in L$ and $r = b_1^{\beta_1} b_2^{\beta_2} \cdots b_s^{\beta_s} \in R$, define $\phi(\ell) = (\alpha_i) \quad \text{and} \quad \phi(r) = (\beta_i)$.
L is adjacent to $\{R\ell \mid \ell \in L\}$; Lr is adjacent to

$$\{R\ell[\ell, r] \mid \ell \in L\} = \{R\ell z_1^{\phi(\ell)\phi(r)^{\mathrm{T}}} z_2^{\phi(\ell)\mathrm{M}\phi(r)^{\mathrm{T}}} \mid \ell \in L\}.$$

Then $X_1 \cong X(s,p,\varphi(x))$ by the map ψ:

$$\psi(Lrz_1^i z_2^j) \;=\; (\phi(r), (i,j)), \quad \psi(R\ell z_1^i z_2^j) \;=\; (\phi(\ell)', (i,j)),$$

where $r \in R$, $\ell \in L$ and $z_1^i z_2^j \in K$.

Step 7: Show that for $X(s, p, \varphi(x))$, its fibre-preserving automorphism group acts 2-arc-transitively.

For $Y = K_{p^s,p^s}$, let $T_1 \cong T_2 \cong \mathbb{Z}_p^s$ such that T_1 (resp. T_2) translates the vectors in U (resp. W) and fixes W (resp. U) pointwise.

(i) Clearly, for the graph $X(s, p, \varphi(x))$, both T_1 and T_2 lifts.

(ii) V is a space over $L = \mathbb{F}_p(\mathrm{M})$, where $\mathrm{M} = \mathrm{B}$.

Let $\mathcal{C}$ be the centralizer of L^* in $\mathrm{GL}(s, p)$. Then $L^* \leq \mathcal{C}$ and for any $c \in \mathcal{C}$, $\ell \in L$ and $v \in V$, we have $(v\ell)c = (vc)\ell$, that is c induces a linear transformation on the L-space V. Therefore, $\mathcal{C} \leq \mathrm{GL}(V, L) \cong \mathrm{GL}(\frac{s}{d}, |L|)$. In particular, $\mathcal{C}$ is transitive on V ($\mathcal{C}$ contains a Single-subgroup).

For any $\mathrm{P} \in \mathcal{C}$, define a map ρ_P on $V(Y)$ by

$$\alpha^{\rho_\mathrm{P}} = \alpha \mathrm{P}^\tau \quad \text{and} \quad (\alpha')^{\rho_\mathrm{P}} = (\alpha \mathrm{P})'$$

for any $\alpha \in V(s, p)$, where τ denotes the inverse transpose automorphism of $\mathrm{GL}(s, p)$. Set

$$H := \langle \rho_\mathrm{P} \bigm| \mathrm{P} \in \mathcal{C} \rangle \leq \mathsf{Aut}\,(Y).$$

Then $H \cong \mathcal{C}$ and H acts transitively on nonzero vectors on both biparts of Y.

For any $\rho_{\mathrm{P}} \in H$, we have

$$\begin{aligned} f_{\alpha^{\rho_{\mathrm{P}}},(\beta')^{\rho_{\mathrm{P}}}} &= f_{\alpha\mathrm{P}^{\tau},(\beta\mathrm{P})'} = (\beta\mathrm{P}(\alpha\mathrm{P}^{\tau})^{\mathrm{T}}, \beta\mathrm{P}\mathrm{M}(\alpha\mathrm{P}^{\tau})^{\mathrm{T}}) \\ &= (\beta\alpha^{\mathrm{T}}, \beta\mathrm{P}\mathrm{M}\mathrm{P}^{-1}\alpha^{\mathrm{T}}) = (\beta\alpha^{\mathrm{T}}, \beta\mathrm{M}\alpha^{\mathrm{T}}) = f_{\alpha,\beta'}. \end{aligned}$$

Thus, we get $f_{W^{\rho_{\mathrm{P}}}} = f_W$ for any closed walk W in Y. By Lifting Theorem, ρ_{P} lifts and so H lifts.

(iii) Take a matrix Q such that $\mathrm{QMQ}^{-1} = \mathrm{M}^{\mathrm{T}}$. Define $\sigma \in \mathsf{Aut}\,(Y)$: $\alpha^{\sigma} = (\alpha \mathrm{Q})'$ and $\beta'^{\sigma} = \beta \mathrm{Q}^{\tau}$ for any $\alpha, \beta \in V(s,p)$. Then

$$\begin{aligned} f_{\alpha^{\sigma},(\beta')^{\sigma}} &= f_{(\alpha \mathrm{Q})', \beta \mathrm{Q}^{\tau}} = -f_{\beta \mathrm{Q}^{\tau},(\alpha \mathrm{Q})'} \\ &= -(\alpha \mathrm{Q}(\beta \mathrm{Q}^{\tau})^{\mathrm{T}}, \alpha \mathrm{QM}(\beta \mathrm{Q}^{\tau})^{\mathrm{T}}) \\ &= -(\alpha \beta^{\mathrm{T}}, \alpha \mathrm{QMQ}^{-1} \beta^{\mathrm{T}}) = -(\beta \alpha^{\mathrm{T}}, \alpha \mathrm{M}^{\mathrm{T}} \beta^{\mathrm{T}}) \\ &= -(\beta \alpha^{\mathrm{T}}, \beta \mathrm{M} \alpha^{\mathrm{T}}) = -f_{\alpha,\beta'}. \end{aligned}$$

Thus, $f_{W^{\sigma}} = -f_W$ for any closed walk W. So σ lifts.

(iv) Check:

$$(t_\alpha)_1^\sigma = (t_{\alpha\mathrm{Q}})_2 \in T_2, \quad (t_\beta)_2^\sigma = (t_{\beta\mathrm{Q}^\tau})_1 \in T_1, \quad (\rho_\mathrm{P})^\sigma = \rho_{\mathrm{Q}^{-1}\mathrm{P}^\tau\mathrm{Q}}.$$

Set

$$A := ((T_1 \times T_2) \rtimes H)\langle\sigma\rangle \leq \mathrm{Aut}\,(Y).$$

Then, A acts 2-arc-transitively on Y. By (i)-(iii), we know that A lifts so that the fibre-preserving automorphism group of the graph $X(s, p, \varphi(x))$ acts 2-arc-transitively. □

Examples 5.4: Application of permutation modules of groups

Permutation Module:
G=trans group on Ω, V=F-space with the base Ω
G-permutation module V induced by natural action of G
X=repres, $X(g)$ is permutation matrix, $\chi(g)$=numbet of fixed points of g.

Theorem: Let $H = G_\alpha$. T=1-repers of H. Then $T^G = KG \bigotimes T$ is the permutation reper.

Theorem: (1) rank $r(G) = [\chi, \chi]$, where $F = C$
(2) G is 2-tran iff $\chi = 1 + \phi$, where $\phi \in Irr(G)$.

Theorem: Let G be a primitive group of degree n and rank r on Ω, and let π be the permutation character of G associated with its action on Ω. Assume that $\pi = \chi_0 + \Sigma_{i=1}^{s-1} e_i \chi_i$, where χ_0 is the principal character, χ_i is the irreducible constituent of degree f_i, and e_i is the multiplicity of χ_i. Then
(1) $r = 1 + \Sigma_{i=1}^{s-1} e_i^2$ and $1 + \Sigma_{i=1}^{s-1} e_i f_i = n$.
(2) If $r \leq 5$, then π is multiplicity-free.
(3) The suborbits of G are all self-paired if and only if π is multiplicity-free and every irreducible constituent χ_i is real-valued.

Permutation Modules of 2-trans groups:

1. Brain Mortimer, The modular of permu repres of the 2-tran groups, *Proc. London Math. Soc* (3) 41(1980), 1-20

Examples of Permutation Modules

$G/K = AGL(3,2) \cong Z_2^3 \rtimes \mathrm{GL}(3,2)$, where $K = Z_2^3$

Special case: $G = Z_2^6 \rtimes PSL(2,7)$

$T := PSL(2,7) \leq \mathrm{GL}(6,2)$,

$\Omega = \mathrm{PG}(2,2)$;

$V = V(\Omega)$: the characteristic functions $\chi(\Delta)$, $\Delta \in P(\Omega)$;

V is a 7-dimensional $\mathrm{PSL}(2,7)$-module by natural action;

V_1: the subspaces of V generated by $\{i, j, i+j \mid i, j \in \Omega, i \neq j\}$

$I = \chi(\Omega)$

$Y = K_8$, $V(Y) = V(3,2)$, $K = (V_1/I, +)$

The cover $K_8 \times_f K$ as follows: $f_{0,j} = \overline{0} := I$

$f_{i,j} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + I$ for all $i \neq j$ in Ω.

Example

$Y = K_{8,8}$, $G \cong \mathrm{AGL}(3,2)$, $K = \mathbb{Z}_r^2$. *Show there exists no cover.*

Proof Let $K = \langle z_1, z_2 \rangle \cong \mathbb{Z}_r^2$, $G \cong \mathrm{AGL}(3,2)$ and $A = G \rtimes \mathbb{Z}_2$. Write $G = T \rtimes H$, where $T \cong \mathbb{Z}_2^3$ and $H = G_u \cong \mathrm{GL}(3,2)$ for some $u \in V(Y)$. Let $\widetilde{G}$ be the lift of G so that $\widetilde{G}/K = G$.

Step 1: Show $C_A(T) \cong \mathbb{Z}_2^4$.

From $A/C_A(T) \leq \mathrm{GL}(3,2)$ we know that $|C_A(T) : T| = 2$ and $C_A(T)$ is abelian, where $C_A(T)$ is isomorphic to either $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^2 \times \mathbb{Z}_4$. Suppose that $C_A(T) \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4$. Take $\sigma \in C_A(T) \setminus T$ such that $|\sigma| = 4$. Then, $\langle \sigma^2 \rangle$ is characteristic in $C_A(T)$ so that it is normal in A. Thus, $\sigma^2 \in Z(A) \cap T = 1$, a contradiction. Therefore, $C_A(T) \cong \mathbb{Z}_2^4$ so that $A = G \rtimes \langle \sigma \rangle$ and Y is a Cayley graph of the elementary abelian group $\mathbb{Z}_2^4$.

Step 2: Show $K \leq Z(\widetilde{G})$.

Let $\widetilde{T}$, τ and $\widetilde{A}$ be the respective lifts of T, σ and A so that $\widetilde{T}/K = T$ and $\widetilde{A} = \widetilde{G}\langle\tau\rangle$, where $\tau^2 \in K$. Then $\widetilde{G} = \widetilde{T} \rtimes \widetilde{G}_{\widetilde{u}}$ for some $\widetilde{u} \in V(X)$. Since

$$C_{\widetilde{G}}(K)/K \lhd \widetilde{G}/K \cong \mathbb{Z}_2^3 \rtimes \mathrm{GL}(3,2),$$

we get $C_{\widetilde{G}}(K)/K \cong 1, \mathbb{Z}_2^3$ or $\widetilde{G}/K$. Since $\widetilde{G}/C_{\widetilde{G}}(K) \leq \mathrm{Aut}\,(K) \cong \mathrm{GL}(2,r)$ and since $\mathrm{GL}(2,r)$ contains no nonabelian simple subgroup, we get $\widetilde{G} = C_{\widetilde{G}}(K)$, namely $K \leq Z(\widetilde{G})$.

Step 3: Show $r = 2$.

Suppose that $r \neq 2$. Let F be a fibre and take a vertex $\widetilde{v} \in F$. Then, $\widetilde{G}_F = K \times \widetilde{G}_{\widetilde{v}}$. Since $(|\widetilde{G} : \widetilde{G}_F|, |K|) = (8, r^2) = 1$ and K is an abelian normal subgroup of $\widetilde{G}$, by Proposition **??**, K has a complement in $\widetilde{G}$, say S. Thus, $\widetilde{G} = K \times S$, where $S \cong G$. For any $\mathrm{GL}(3,2) \cong L \leq \widetilde{G}$, since $L \cap S \trianglerighteq L$, we have $L \cap S = 1$ or L. If $L \cap S = 1$, then $L \cong LS/S \leq KS/S \cong K$, a contradiction. So $L \leq S$. Thus, for an edge $\widetilde{u}\widetilde{w} \in E(X)$, both $\widetilde{G}_{\widetilde{u}}$ and $\widetilde{G}_{\widetilde{w}}$ are contained in S so that $\langle \widetilde{G}_{\widetilde{u}}, \widetilde{G}_{\widetilde{w}} \rangle \leq S \neq \widetilde{G}$, which follows that X is disconnected. Therefore, $r = 2$.

Step 4: Show $\widetilde{T} \cong \mathbb{Z}_2^5$.

Suppose that $\widetilde{T}/K = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle \cong \mathbb{Z}_2^3$. Taking into account, $\widetilde{G} = \widetilde{T} \rtimes \widetilde{H}$, where $\widetilde{H} := \widetilde{G}_{\widetilde{u}} \cong \mathrm{GL}(3,2)$, acting 2-transitively on $\widetilde{T}/K = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3 \rangle \setminus \{\overline{1}\}$ by conjugacy. Then $\widetilde{H}$ acts transitively on

$$\{[t, t'] \mid t, t' \in \widetilde{T} \setminus K, tK \neq t'K\}.$$

Take h such that $[x_1, x_2]^h = [x_1, x_3]$. Since $\widetilde{T}' \leq K \leq Z(\widetilde{G})$, we get $[x_1, x_2] = [x_1, x_3]$. Similarly, we have $[x_1, x_2] = [x_2, x_3] = [x_1, x_2 x_3]$. However, $[x_1, x_2 x_3] = [x_1, x_2][x_1, x_3] = [x_1, x_3]^2 = 1$ and then $[x_1, x_2] = [x_2, x_3] = 1$. Therefore, $\widetilde{T}$ is abelian. Further, let $g \in \widetilde{H}$ such that $\overline{x}_1^g = \overline{x}_1 \overline{x}_2$. Then, we have $x_1^g = x_1 x_2 k$ for some $k \in K$, which deduces that $x_1^2 = (x_1^2)^g = x_1^2 x_2^2 k^2 = x_1^2 x_2^2$ so that $x_2^2 = 1$. With the same discussion as above, one may get $x_1^2 = x_3^2 = 1$. Therefore, $\widetilde{T} = \langle x_1, x_2, x_3, z_1, z_2 \rangle \cong \mathbb{Z}_2^5$.

Step 5: Show $\widetilde{T}\langle\tau\rangle \cong \mathbb{Z}_2^6$.

Recall that τ is a lift of $\sigma \in C_A(T)$. Since $\widetilde{T}\langle\tau\rangle$ acts regularly on $V(X)$, we may assume that $X \cong \mathbf{Cay}(\widetilde{T}\langle\tau\rangle, S)$ for some subset S of $\widetilde{T}\langle\tau\rangle$. Write $S := \langle \tau\ell_i | \ell_i \in \widetilde{T}, 1 \le i \le 8\rangle$.

As X is undirected, we have $S^{-1} = S$, that is,

$$(\tau\ell_i)^{-1} = (\tau\ell_i)^{-2}\tau\ell_i \in S,$$

where $1 \le i \le 8$. Since $(\overline{\tau\ell_i})^2 = \overline{1}$, we get $(\tau\ell_i)^{-2} \in K$. Suppose that $k = (\tau\ell_i)^{-2} \ne 1$. Then the vertex 1 in $\mathbf{Cay}(\widetilde{T}\langle\tau\rangle, S)$ is adjacent to two different vertices $(\tau\ell_i)^{-1} = k\tau\ell_i$ and $\tau\ell_i$, which are contained in the same fibre, a contradiction. So $k = (\tau\ell_i)^{-2} = 1$, that is

$$(\tau\ell_i)^2 = 1 \quad \text{for any} \quad 1 \le i \le 8. \tag{3}$$

From $1 = (\tau\ell_i)^2 = \tau^{-2}\ell_i^\tau \ell_i$, we get

$$\ell_i^\tau = \tau^2 \ell_i \quad \text{for any} \quad 1 \leq i \leq 8, \tag{4}$$

recalling $\tau^2 \in K$.
By Eq(4) and $\tau^2 \in K$, we get

$$[\tau\ell_i, \tau\ell_j] = \ell_i \tau^{-1} \ell_j \tau^{-1} \tau \ell_i \tau \ell_j = \ell_i \ell_i^\tau \ell_j^\tau \ell_j = \ell_i \tau^2 \ell_i \tau^2 \ell_j \ell_j = 1, \tag{5}$$

for any $1 \leq i, j \leq 8$.
Since X is connected, we get $\widetilde{T}\langle\tau\rangle = \langle S\rangle = \langle \tau\ell_i | 1 \leq i \leq 8\rangle$, which is an elementary abelian group by Eq(3) and Eq(5).

Now, since $\widetilde{T} \times \langle \tau \rangle \cong \mathbb{Z}_2^6$, which acts regularly on X, we may identify $V(X)$ with $V(6,2)$. Let $\widetilde{H} = \widetilde{A}_0$, where $0 \in V(X) = V(6,2)$. Then, $\widetilde{H}$ acts 2-transitively on the neighborhood $S = X_1(0)$ of 0 with cardinality 8. As X is connected, $\widetilde{T} \times \langle \tau \rangle$ is generated by S. Since $C_A(T) \lhd A$, we know that $(\widetilde{T} \times \langle \tau \rangle)/K \lhd \widetilde{A}/K$. Thus, $\widetilde{T} \times \langle \tau \rangle \lhd \widetilde{A}$ so that

$$\widetilde{A} = (\widetilde{T} \times \langle \tau \rangle) \rtimes \widetilde{H} \cong \mathbb{Z}_2^6 \rtimes \mathrm{PSL}(2,7).$$

For S, let $V = V(S)$ be the corresponding permutation $\widetilde{H}$-module.

Also, consider $\widetilde{T} \times \langle \tau \rangle = \mathbb{Z}_2^6$ as an $\widetilde{H}$-module in the conjugacy action.

Thus we get two $\widetilde{H}$-modules, that is, the 8-dimensional module V and the 6-dimensional module $\widetilde{T} \times \langle \tau \rangle$. Furthermore, define a map $\phi : V \to \widetilde{T} \times \langle \tau \rangle$ by the rule

$$\sum_{i \in S} k_i \chi_{\{i\}} \mapsto \sum_{i \in S} k_i i, \quad k_i \in \mathbb{Z}_2.$$

Obviously, ϕ is an $\widetilde{H}$-module epimorphism, where the kernel $\mathrm{Ker}\phi$ should be an $\widetilde{H}$-module with dimension 2. Note that $\mathrm{Ker}\phi$ contains four elements, we know that $\widetilde{H} \cong \mathrm{PSL}(2, 7)$ acts trivially on $\mathrm{Ker}\phi$. Then H fixes at least three 1 dimensional subspaces. However, it was proved that $\langle \mathbf{1} \rangle$ is the only 1-dimensional $\widetilde{H}$-subspace, where $\mathbf{1}$ denotes the constant function. Therefore, our covering graph X does not exist. □

1. Elementary covers:

$G = Z_p^k.T$, where T may be simple group, affine group and so on, depending on your base graph. If we want to go further in this problem, we have to understand more from the related tools, such as

(1). Classical group extension theory, Central extension theory, as well as Cohomology Theory

(2). p-group theory

(3). (ordinary and modular)-representation theory, permutation modular (in particular, that of 2-tran groups) theory

(4). Investigate new methods from different branches.

2. Covers of regular maps:

Study the covers of regular maps.

Gareth Jones Classified all elementary abelian Z_p^k regular covers of platonic maps (ordinary case, $p \neq 2, 3 \quad 5$). The modular case is still in preparation.

End

Thank You Very Much !